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AN INFEASIBLE PRIMAL-DUAL INTERIOR POINT METHOD AS APPLIED TO THE STATIC TRANSMISSION EXPANSION PLANNING PROBLEM

THESIS

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BY:

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Before Starting

George Dantzig presented the linear programming model and the simplex method for solving the problem at an econometrics conference in Wisconsin in the late 40s. The economist Hotelling stood up, devastatingly smiling, and stated that “But we all know the world is nonlinear.” The young graduate student George Dantzig could not respond, but was defended by John Von Neumann, who stood up and concluded that “The speaker titled his talk ‘linear programming’ and carefully stated his axioms. If you have an application that satisfies the axioms, well use it. If it does not, then don’t”; he sat down, and Hotelling was silenced. (See Dantzig’s account of the early history of linear programming in Lenstra, J. K., Rinnooy Kan, A. H. G. and Schrijver, A. eds., *History of mathematical programming*. A Collection of Personal Reminiscences, North-Holland, pp. 19-31, Amsterdam, 1991.)

> *Nothing in the world takes place without optimization,*
> *and there is no doubt that all aspects of the world*
> *that have a rational basis can be explained*
> *by optimization methods.*
>
> *Leonhard Euler, 1744.*

Abstract

A solution for the Static Transmission Expansion Planning (STEP) problem with an Interior Point Method (IPM) considering convex and continuous relaxation modeling is addressed in this work.

The STEP is a large-scale hard combinatorial optimization problem, with the aim of deciding, optimally, the best circuit addition configuration in a power system for a fixed future time. This is a very challenging task because it has a nonlinear, nonconvex and mixed formulation. However, in order to get a more tractable problem, the classical convex relaxation in the network representation by the transportation model is used in this research.

In order to solve the mixed-integer problem, we follow the next strategy. First, the transportation model is used, but with no integer variable constraints. This leads to a continuous relaxed Linear Program (LP) which has to be solved in every iteration of the expansion planning process. In this case, an infeasible logarithmic barrier primal-dual IPM is applied as a subroutine to solve the resulting LP. The interest in using this kind of methods, is due to their efficiency for solving large-scale problems (the STEP formulation is generally a large-scale problem). Finally, the Garver's Constructive Heuristic Algorithm (CHA) is used to obtain solutions that satisfy the dropped integer constraints.

For these purposes, a tool in MATLAB R2013a environment was developed, where the CHA and the IPM were completely coded.

The thesis includes results for the classical 6-bus Garver's test system and for the 24-bus IEEE test system.

Resumen

En este trabajo se presenta la solución del problema de la Planeación de la Expansión de la Transmisión Estática (PETE) considerando una relajación convexa y continua en el modelado.

La PETE es un problema combinatorio muy complicado y de gran escala que busca decidir de forma óptima la mejor configuración de adición de circuitos en un sistema de potencia para una fecha fija futura, el cual sigue siendo un gran reto debido a que su formulación es no-lineal, no-convexa y mixta. Sin embargo, con la finalidad de lidiar con un problema más tratable, en este trabajo se usa la clásica relajación convexa para representar la red vía el modelo de transporte.

Para resolver el problema lineal-entero, se llevó a cabo la siguiente estrategia. Primero, se usó el modelo de transporte pero sin las restricciones para las variables enteras. Esto genera un programa lineal continuo que debe ser resuelto en cada iteración del proceso de la planeación de la expansión. En este caso, se aplicó un método infactible primal-dual de barrera logarítmica de puntos interiores como subrutina para resolver el programa lineal que resulta. El interés de usar este tipo de métodos se debe a la eficiencia que presentan al resolver problemas de gran escala (la formulación de la PETE es en general un problema de gran escala). Finalmente, con el fin de obtener soluciones que satisfagan las restricciones omitidas sobre las variables enteras, se usa el conocido algoritmo heurístico constructivo de Garver.

Para estos fines, se desarrolló una herramienta en MATLAB R2013a, donde se programaron completamente tanto el algoritmo heurístico constructivo como el método de puntos interiores.

La tesis incluye resultados del sistema clásico de prueba de 6-nodos de Garver y del sistema de prueba de 24-nodos de la IEEE.

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Dedications

To my wife Pilar
For all her love, understanding and unconditional support.
I love you beauty!

Your smile and happiness
are my greatest inspiration.

To my little ones, Ian and Andrea.
As a small sample of perseverance and work.

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Nomenclature

Acronyms

AC	Alternating Current
CHA	Constructive Heuristic Algorithm
DC	Direct Current
IPM	Interior Point Method(s)
KCL	Kirchhoff Current Law
KKT	Karush-Kuhn-Tucker
KVL	Kirchhoff Voltage Law
LP	Linear Program / Linear Programming
MINLP	Mixed Integer Non-Linear Program / Mixed Integer Non-Linear Programming
OPF	Optimal Power Flow
PDIPM	Primal-Dual Interior Point Method(s)
PSAT	Power System Analysis Toolbox
ROW	Right Of Way
SI	Sensitivity Index
STEP	Static Transmission Expansion Planning
WOR	WithOut Redispatch
WR	With Redispatch

Notation

n_b	Number of buses
n_g	Number of generation buses
n_R	Number of ROW's
v	Cost of construction of transmission circuits (objective function value)
v_1	Primal stopping criterion
v_2	Dual stopping criterion
v_3	Complementarity stopping criterion

Notation

ε_1	Tolerance for the primal stopping criterion
ε_2	Tolerance for the dual stopping criterion
ε_3	Tolerance for the complementarity stopping criterion
ε_μ	Tolerance for the $\mu-$stopping criterion
ϵ	Cost of construction stopping criterion / Stopping parameter for the CHA algorithm
α	Step length
σ	Centering parameter
ρ	Complementarity gap
μ	Barrier parameter

Indexes

i, k	Nodes in the system
l	Transmission lines
l_0	Shunt element of a transmission line
p	Number of equality constraints
q	Number of inequality constraints

Sets

$\mathcal{C}$	Central path
C^n	Set of the n-continuously differentiable functions
$\mathcal{F}_p$	Feasible set for the primal problem
$\mathcal{F}_d$	Feasible set for the dual problem
$\mathcal{F}$	Primal-dual feasible set
$\mathcal{K}$	Set of all buses directly connected to a specific bus
$\mathcal{O}$	Set of all ROW's in the system
$\mathbb{R}^n$	Real n-dimensional Euclidean space
Ω	Feasible set in a LP

Vectors and Matrices

$c \in \mathbb{R}^{n_R}$	Cost of construction vector
$d \in \mathbb{R}^{p}$	Vector of demand
$e \in \mathbb{R}^{n}$	Vector of all ones of appropriate dimension, $e = [1, 1, \ldots, 1]^T$
$f \in \mathbb{R}^{n_R}$	Vector of power flows
$\overline{f} \in \mathbb{R}^{n_R}$	Vector of maximum power flows
$g \in \mathbb{R}^{n_g}$	Vector of bus generation
$\overline{g} \in \mathbb{R}^{n_g}$	Vector of maximum bus generation
$n \in \mathbb{R}^{n_R}$	Vector of candidate transmission circuits
$\overline{n} \in \mathbb{R}^{n_R}$	Vector of the maximum number for transmission circuits construction
$n^0 \in \mathbb{R}^{n_R}$	Vector of initial transmission circuits in ROW
$s \in \mathbb{R}^{q}$	Slack variables vector (primal slacks)
w	Vector containing all vectors of variables; $w := [x, s, \lambda, \pi]$
x	Vector of variables for the STEP formulation; $x := [n, f, g]$
$\lambda \in \mathbb{R}^{p}$	Lagrange multipliers vector for the equality constraints (dual slacks)
$\pi \in \mathbb{R}^{q}$	Lagrange multipliers vector for the inequality constraints (dual slacks)
S	Diagonal matrix with $s_i \in s$ elements
Π	Diagonal matrix with $\pi_i \in \pi$ elements
$\mathcal{S}$	Node-branch incidence matrix

Constants

$c_{ik} \in c$	Cost of construction of one circuit in the ROW (i, k)
$d_i \in d$	Demand in the $i - th$ bus
$\overline{f}_{ik} \in \overline{f}$	Maximum power flow in the ROW (i, k)
$\overline{g}_i \in \overline{g}$	Maximum generation in the $i - th$ bus
$\overline{n}_{ik} \in \overline{n}$	Maximum number for transmission circuits construction in the ROW (i, k)
$n^0_{ik} \in n^0$	Initial transmission circuits in ROW (i, k)
w^0	Starting solution for the vector w, *i.e.*, $[x^0, s^0, \lambda^0, \pi^0]$
α_0	Constant for the step length parameter
ϵ	Cost of construction stopping criterion
σ	Centering parameter
μ^0	Initial barrier parameter

Variables

$f_{ik} \in f$	Power flow in the ROW (i, k)
$g_i \in g$	Generation in the $i-th$ bus
$n_{ik} \in n$	Candidate transmission circuit in the ROW (i, k)
$s_i \in s$	Slack variables (primal slacks)
$\lambda_i \in \lambda$	Lagrange multipliers for the equality constraints (dual slacks)
$\pi_i \in \pi$	Lagrange multipliers for the inequality constraints (dual slacks)

Chapter 1

Introduction

This chapter is devoted to the general context of this research. First, the theoretical background of STEP is given. Afterwards, the research motivation, the objectives and the literature review of the IPM applications on optimal power flows and the transmission expansion planning problem are exposed. A brief history about the evolution of the linear programming solution techniques is included as well.

Transmission planning is now an essential part of the reliability and security of electricity supply in which a decision on expansion in the past, affects the performance of the power system in the present, as well as decisions made in the present will affect the performance of the power system in the future. As very high economic cost decisions, both, the design of the strategy of the expansion and the construction of the transmission elements must be done for long life infrastructure, providing obsolescence of components and avoiding the waste of financial resources.

STEP has always been a rather complex task in general. For example, the location and capacity of the load that will be integrated into the system are not known with full certainty. Moreover, once the load is installed, we should take into account the power consumption; which also includes the load already installed. This means that the load is a function that depends on both, space and time. The variation of demand over time is solved with an appropriate load dispatch, while the problem of the location and capacity of the load should be considered within the STEP problem.

In general, if a future generation and demand scenario have been given...
"the STEP is a problem of synthesis and decision-making with the aim of getting a plan of construction expansion of transmission elements over a fixed planning horizon, to ensure a power system able to meet the demand under certain standards of quality and reliability, at the lowest possible cost for both investment when building while operation when servicing."

STEP Objective

The purpose of STEP is to determine the location, amount and type of transmission elements that should be added (or even removed) for a fixed future time, which will have proper power transport capacity, capable of withstanding future generation and load additions as well as their flow requirements [1].

It is expected that the new system meets the constraints of the problem with a certain level of quality and reliability at the lowest possible cost that must be able to operate properly at least under N-1 contingency conditions.

Basically, the objective pursued in a static expansion is to decide optimally *where*?, *how many*? and *what*? transmission elements should be built and/or removed. If the expansion is dynamic, the question *when*? to do such maneuvers must be answered.

Solution Methods and Formulations for the STEP Problem

Algorithms for solving the problem of transmission expansion can be classified in [2, 3]: heuristics, mathematical optimization and metaheuristics.

> **Heuristics:** This type of formulations are based on intuitive arguments and rules that mostly use common sense and experience to get a good solution. However, this process does not guarantee finding the optimal solution to any problem. These methods are usually applied when a problem is very difficult to solve [4].
>
> **Mathematical Optimization:** Here, the problem is formulated purely as an optimization model with an objective function which measures the performance of any parameter to be minimized or maximized; subject to a set of constraints that restrict the solution space.
>
> **Metaheuristics:** These algorithms integrate the features of optimization and heuristic methods. Due to the non-linear, non-convex and combinatorial nature of the STEP problem, these methods have been quite used to solve the problem, generating high quality solutions in short computational times.

When applying any method of solution it is necessary to observe [5]:

1. **The uncertainties** in the power system. This will focus the study in a
    (a) deterministic approach
    (b) non-deterministic approach

   When the study is *deterministic*, the expansion plan is designed based on historical data, choosing the worst scenario ever presented where all parameters (generation, load growth, etc.) are given by a single value.

Under a *non-deterministic* approach we have the possibility of considering the past experience and future expectation, since at least one of the parameters is presented by means of a random variable, characterized by a given probability distribution.

2. **Temporal treatment**. This will produce a

    (a) static formulation of the problem

    (b) dynamic formulation

    When we work from a *static* formulation, we should determine an optimal solution for a certain point of time (ideally but not necessarily a short period). Here, the first three questions of the objective of the STEP problem are answered (*where*?, *how many*? and *what*? transmission elements should be installed).

    From a *dynamic* formulation –as the gradual temporal variations of the studied system are considered for the purpose of research– we should obtain a sequence time of the accomplishment of the optimal expansion plan. Since this view, in addition to answering the questions *where*?, *how many*? and *what*? transmission elements should be built and/or eliminated, the planning process generates an optimal response to the question *when*? to do this.

3. **Power system structure**. This allows to know if it is a

    (a) regulated environment

    (b) deregulated environment

    Under a regulated environment a vertically integrated utility is the unique responsible for reaching the mentioned goals of STEP. After deregulation, the transmission system must be able to become a mean to facilitate and promote competition without any advantage to an agent as well, between other goals; for an excellent treatment of the market environment of the transmission expansion you can see [6].

1.1 Research Motivation

The STEP mathematical formulation can be expressed as a Mixed-Integer Nonlinear Programming (MINLP) problem in the following general form:

$$\begin{array}{rl} \text{minimize} & f(x) \\ \text{subject to} & g(x) = 0 \\ & h(x) \geq 0 \end{array} \tag{1.1}$$

where

$x \in \mathbb{Z}^r \times \mathbb{R}^n$: is a vector of decision variables which in the STEP problem could include *continuous variables* such as power generation (active and reactive), bus voltages, bus angles and power flows, and *integer variables* such as tap ratios of controllable transformers and/or circuit additions.

$f : \mathbb{Z}^r \times \mathbb{R}^n \to \mathbb{R}$: is a scalar function that represents the transmission system planning goal such as investment cost of new circuit additions, reliability cost, congestion cost, electricity market cost, etc.

$g : \mathbb{Z}^r \times \mathbb{R}^n \to \mathbb{R}^p$: is a vector of nonlinear functions that contains the AC or DC network representation.

$h : \mathbb{Z}^r \times \mathbb{R}^n \to \mathbb{R}^q$: is a vector of nonlinear functions that includes mandatory power system operational functions such as limits of transmission power in circuits, limits of the generator output power, limits of voltage levels and limits of the maximum number of circuits that can be added, between others. This vector can contain optional constraints such as the limits of the investment, the reliability and security limits, and the environmental impact limits.

Note that if $\mathbb{Z}^r = \emptyset$, the optimization problem (1.1) is not mixed anymore. The couple of index p and q give the number of equality and inequality constraints, respectively; we will consider that total number of constraints will be always $m = p + q$. Note also that for simplicity and without loss of generality, we consider that the variable bounds $\underline{x} \leq x \leq \overline{x}$ are included in $h(x)$.

In the more general sense, solving (1.1) is a very challenging and maybe could be an impossible task to do. Perhaps, the most important drawbacks for the problem are two:

1. *One related to the network representation (AC or DC power flow equations).* The nonlinearity and nonconvexity of both formulations joined to the nonconvexity nature of the feasible region produced, yield a problem with a multimodal landscape. Of course, the chances in the majority of the algorithms to be trapped in a local minimum are large [7]. In fact, great efforts have been made in order to have simpler –but more accurate– models of the network [8, 9, 10, 11, 12].

2. *The mixed nature of the problem.* It is well-known that mixed integer programming problems belong to the class of NP-hard problems [13, 14], which means that they cannot be solved exactly by polynomial time algorithms; you can see [15] for a proof that STEP is a NP-hard problem. Moreover, when one tries solving a nonconvex MINLP problem, there will be no guarantee to obtain the global optimum solution. This topic remains unresolved for the most practical optimization models of complex system [8].

Roughly speaking and once we have already chosen the network representation (AC or DC network equations), we can solve the STEP problem in a direct form or in a relaxed form. Of course, as a very difficult problem and in order to have a more tractable formulation, relaxations of (1.1) are made in general.

Nevertheless the formal definition of relaxation will be given in Chapter 2, we can say for the moment that the relaxation of a hard given problem is another problem which underestimates their objective function and/or their constraints. As a consequence, an optimum value of the relaxed problem is a lower bound of the one of its original problem. Two common relaxations are the *convex relaxation* and the *continuous relaxation*. A relaxation is said to be convex if the objective function and the feasible region are convex. In the case of a mixed problem, the continuous relaxation corresponds to the problem obtained by dropping the integer restrictions [16].

We could use the latter notion to classify the *mathematical modeling approaches* of the STEP as: convex relaxed and continuous relaxed.

When a continuous relaxed modeling is performed, an heuristic to obtain the integer solutions are usually employed; this still being the most reported and worked approach for the STEP [7, 17, 18, 19, 20, 21, 22]. On the other hand, when continuous relaxation is not taken into account in the modeling process, the use of optimization techniques with the capability to solve the problem considering its mixed nature is necessary; methods like Benders decomposition and branch and bound are classical approaches for mixed problems in STEP. Because of the difficulty of the task, only few papers using these methods are reported in the literature, among these papers are [23, 24, 25]. Moreover, the convex relaxation can or cannot be made and will lead to a linear or nonlinear optimization problem inside the STEP formulation, respectively.

1.2 Research Objectives

The state of the art shows that the most demanding and difficult part of the STEP process is solving the optimization problem which appears in the expansion activity. Thus, in this work we strive for the efficient solution of the large-scale optimization problem generated in the process, testing an infeasible IPM which is recognized as the most efficient method for solving this kind of large formulations; the algorithm described in this thesis is based on the first part of the work developed at the Statistics and Operations Research Department of the Princeton University by Vanderbei & Shanno (the algorithm LOQO for nonconvex nonlinear programming) [26].

In this way, and considering the convex and continuous relaxation formulation of the STEP problem, the main objectives of the current thesis research are:

- To formulate in detail the STEP problem as a mixed-integer LP using the transportation model.

- To show the development of an infeasible logarithmic barrier primal-dual interior point method for solving the continuous relaxation version of the transportation model to the STEP problem.
- To show the connection between optimality conditions and IPM through the central path concept.
- To solve the STEP problem using the Garver's constructive heuristic algorithm in order to face the continuous relaxation performed.

Moreover, last but not least, in this work –as one of the main objectives– we have done great efforts at obtaining a formal but at the same time a very understandable document with a complete development of all subjects, giving clarity on how to model the problem, on the mathematics used and on the method of solution as well. All this in order to contribute to continue this line of research.

1.3 Background

Optimal network expansion has always been one of the most important issues in power system planning, and has been studied extensively during the past several decades. The people interested in this problem have investigated the transmission expansion with different objectives and constraints, from different aspects such as modeling and solving methods, and from the electricity market and uncertainty points of view.

For a general review of the transmission expansion planning state of the art, there are four excellent surveys [2, 3, 27, 28] from 2003, 2006, 2013 and 2014, respectively. In this section we will comment about those papers which are directly related to our work only.

Since STEP problem by nature can be regarded as an optimal power flow (OPF) problem with discrete constraints [12], we will begin with a review of the IPM in OPF. Then, we will show the use of the IPM in transmission network expansion planning when convex and/or continuous relaxation is performed. Finally, we give a short review of the works related to STEP in Mexico.

1.3.1 Interior point methods in optimal power flows

The use of IPM in power systems has experienced an awesome expansion as it is shown in [29], where we can find a collection of the different methods and applications of IPM to power systems. In particular –speaking about optimal operation of power systems– OPF is the problem which has been solved most of the times using an IPM, as it is reported in [30].

The use of IPM in OPF has been very successful. For example, Geraldo Leite in his outstanding Ph. D. dissertation [31], proposed and investigated in great detail a number of

IPM for solving some variants of the OPF; here, interior and even noninterior point methods were successfully applied, including the study of the so-called higher-order IPM: the predictor-corrector method (PCM) [32], the multiple predictor-corrector method (MPCM) [33] and the multiple centrality corrections method (MCCM) [34].

The success of the IPM in OPF, encourage to Rider et al. to look for a more robust IPM –with faster convergence– and they have explored a combination of the PCM, MPCM and MCCM [35].

In an excellent paper by Capitanescu et al. in 2007 [36], two higher-order methods –PCM and MCCM– where compared with regard to their performances against the pure PDIPM when applying to the OPF. In 2012, Frank et al. reported that primal-dual methods have demonstrated excellent performance in solving many OPF problem variants [37]. Later, in 2013 Capitanescu and Wehenkel concluded that MCCM was the most reliable IPM for OPF [38]. In [39] Chiang shows efficient approaches to solve the $n-1$ security constrained OPF by using structure-exploiting IPM.

In fact, Leite, Rider et al. and Capitanescu together with Wehenkel verified the IPM efficiency when working on very large power systems, respectively 2098, 2256 and 8387 buses in their works. Being the first two models part of the Brazilian interconnected system, and the latter the model of a large part of the interconnected EHV European power system.

1.3.2 Interior point methods in transmission planning

The use of IPM in power systems has become very important because of its advantages when large (or very large) optimization problems arise, as it does in transmission planning.

The seminal work of Garver let a clear understanding that when continuous relaxation is done, it is necessary to solve large LP or NLP subproblems in the transmission planning process [40]. This has prompted researchers to look for better solvers which can help in addressing the subproblem.

In 2005, Sánchez et al. presented an IPM as an innovative subroutine for solving the LP [20]. They performed a convex relaxation –through the transportation model– and a continuous relaxation as well. Results showed that IPM as a solver inside a CHA found good quality solutions for medium scale systems (for instance the 46-bus South Brazilian system and the 93-bus Colombian system).

The approach of continuous relaxation allowed the use not only of linear models for the network, but also the use of more complicated models as the DC model in [21], the AC model in [19] and the AC model including reliability constraints in [17]. In these cases the IPM was as the nonlinear subproblem solver. In Correa et al. [41], IPM were used to solve linear and non-linear formulation of their environmental model where the emissions of CO_2 were considered.

When the continuous relaxation is not applied, the IPM still play an important role. In this case, the STEP problem has a more difficult mixed formulation and in such case, it is possible to do a convex relaxation through the transportation model in order to solve the problem as in [24, 25] where a branch-and-bound algorithm was used.

1.3.3 The Mexico's context

It is known that the problem of the expansion planning of transmission systems is undoubtedly a very important task in power systems, and in the *electrical research and postgraduate studies section* at the ESIME has not gone unnoticed [42, 43, 44]. For example, nonetheless Molina in [42] did not work on the expansion planning problem itself but on a post-problem issue (the transmission oeuvre prioritization), he introduced for the first time these topics in the postgraduate section. More recently, Díaz [43] developed a tool for the transmission expansion considering deregulation and Anaya in [44] worked on the problem of the network expansion including the effects of wind production penetration.

At the UNAM, Zenón worked on a hybrid mechanism for the network expansion in Mexico, the United States and Canada [45].

On the other hand, the National Grid in Mexico is formed by a transmission system based on 400 kV, 230 kV and 115 kV lines and since the early 60's (where the nationalization of the electricity industry in Mexico took place) until the energetic reform in January 2016, the transmission system expansion and planning was centrally performed by the Federal Electricity Commission (Comisión Federal de Electricidad, CFE) using public resources, taking into account their own necessities and subject to financial time constraints.

According to Madrigal [46] and to Ávila and Mota [47], the procedures and methodologies for transmission management and expansion planning in Mexico from the centralized point of view of the CFE was based on a minimum cost analysis aimed at minimizing the expected investment and operational costs, subject to technical and economic constraints where the location, sizes and dates for the hydroelectric and thermal plants are determined at the beginning of the process by the planning division in central offices of the CFE. This procedure selects projects which show long-term utilization, that improve system reliability and are least cost options. The planning methodology also includes a profit analysis which quantifies costs and benefits of the transmission program [46].

After the unbundling of the generation, network activities (transmission and distribution) and retailing segments in the Mexican Electricity Market (where competition in generation and retailing segments is now allowed), the expansion and upgrading activities of the National Transmission Network is proposed by the National Control Energy Center (Centro Nacional de Control de Energía, CENACE), considering the opinion of the Energy Regulatory Commission (Comisión Reguladora de Energía, CRE) and –in a last stage– authorized by the Energy Ministry (Secretaría de Energía, SENER).

After deregulation, the objective of the transmission network is to facilitate the competition being a fair field where all participants of the market can have the same opportunities and where the market operator can decide for the best option for the consumers.

1.4 Why to Use an Interior Point Method for a Linear Programming Problem? A brief history

Linear programming has its roots in the work of Fourier in 1826 in his study of linear inequalities. The applied side of the subject got its start in 1939 when L. V. Kantorovich noted the practical importance of a certain class of linear programming problems and gave an algorithm for their solution; the work was unknown in the West until 1960 when the English version appeared [48]. Actually, in 1975, the Royal Swedish Academy of Sciences awarded the Nobel Prize in economic science to L.V. Kantorovich and T.C. Koopmans "for their contributions to the theory of optimum allocation of resources".

In a general sense, a LP is a combinatorial problem which selects an extreme point among a finite set of possible vertices in a polyhedron (which models the feasible region of the problem defined by the constraints). In 1947, George Dantzig presented the first algorithm that performs in a systematic way that selection. The algorithm is known as *simplex method*. In his work, Dantzig presented the solution for the problem of finding the best assignment of 70 people to 70 jobs modeled as a LP [49].

Since its beginning and for about 25 years, the simplex method was evolving and a lot of variants of the method were developed as codes for solving in a very efficient way large LP. Besides, it is known that the optimal solution of a LP always lies at a vertex of the feasible region, and that the simplex method proceeds from one vertex to a neighboring vertex until it hits the optimal one. This could be an inconvenient topic for certain types of problems as Victor Klee and George Minty showed in their work of 1972 [50]. Klee and Minty proved that, in the worst case, the method has exponential complexity in the size of the problem, *i.e.*, the method needs an exponential number of iterations to find the optimal solution[1]. In their problem –with n variables, n restrictions and 2^n vertex– they showed that the algorithm must visited *every* vertex before reach the optimal solution (for a version of the Klee and Minty problem, see [52]). Although this rarely happens in practice, this was a detonating question and people began to look for another linear programming algorithm with a polynomial complexity, this means, an algorithm in which the running time required to compute the solution should be bounded above by a polynomial in the size, or the total data length, of the problem.

In 1979, based on an ellipsoid method developed by other Russian mathematicians, Khachiyan presented a polynomial algorithm for LP [53]. Khachiyan showed that his algorithm had a polynomial complexity of order $(nm^3 + m^4)L$, where m represents the number of rows in a LP formulation, n the number of columns, and L the length of the data [54]. Of course this had a great attention even on the international press because of the great theoretical advance. However, it was quickly a big disappointment because practitioners realized that the best implementation of the ellipsoid method was not com-

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[1] The term *complexity* refers to the amount of resources required by a computation. For an excellent treatment of the LP complexity you can see [51].

petitive with the simplex method; the number of steps required for the ellipsoid method to terminate was almost always close to the worst case bound –whose value, although defined by a polynomial, is very large– in contrast to the number of steps for the simplex method which seems to be roughly linear in m and perhaps logarithmic in n [55].

The last contradiction was solved by Karmakar [56], whose announcement in 1984 of a new polynomial-time algorithm for LP with the potential to dramatically improve the practical effectiveness of the simplex method –reporting solution times up to 50 times faster than this one– made front-pages news in major newspapers and magazines throughout the world. The complexity of the Karmakar's method is of order $(nm^2 + m^3)L$ [54].

Karmakar's method belongs to a class of methods called *interior-point methods.* This kind of methods have been demonstrated to be competitive with the simplex method and usually superior on very large problems.

1.4.1 Simplex and interior point methods overview

Every LP is based on the *Fundamental Theorem of Linear Programming* which ensures that the optimum –if it exists– will be at a vertex of the polyhedron formed by the set of constraints (Appendix A). According to this, LP solution methods differ basically in the way of searching that vertex. In fact, the simplex was the first proposal which implemented an intelligent search of the optimum.

There are some differences between the simplex method and IPM that can be easily understood from the geometrical point of view.

The simplex algorithm works roughly as follows [57]: We begin with a feasible point at one of the vertices of the polyhedron. Then we "walk" along the edges of the polyhedron from vertex to vertex, in such a way that the value of the objective function monotonically decreases at each step. When we reach a point in which the objective value decreases no more, we will be finished.

On the other hand, the IPM solves a LP by generating a sequence of points which are inside of the feasible region starting from an initial (strictly) interior point. This means that an IPM starts and moves always in the interior of the feasible region towards the optimum.

From the latter, it is clear that contrary to the simplex method, an IPM never gives exact optimal solution; instead, it generates an infinite sequence converging towards an optimal solution. Of course, after a finite number of iterations it is necessary a stopping scheme for the IPM.

In the Figure 1.1, the searching of the optimum with the simplex method $(x_0^s, x_1^s, \ldots, x_n^s)$ and with IPM $(x_0, x_1, \ldots, x_n)$ is shown.

Each interior point iteration is expensive to compute but can make significant progress towards the solution, while the simplex method usually requires a larger number of inexpensive iterations.

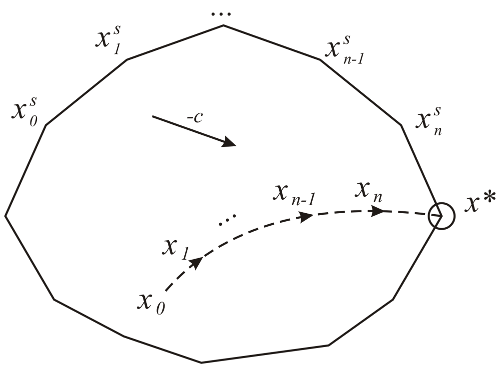

Figure 1.1: Simplex vs. IPM: Geometrical behavior when searching the optimum.

Instead of Simplex Methods where the founding concept are *basis*, in IPM the notion of *central path* and *analytic center* are vital.

1.5 Contributions

The author strongly believes that the main contribution of this thesis is the explanation and exposure in detail of the transmission expansion planning problem, its related mathematics and its solution process using a CHA. As it can be seen along the work, all the topics were developed in a very comprehensive manner and in such a way that a future extension for more complicated models can be straightforward.

Added to this, another important contributions are:

- A clear classification of the STEP problem according to the mathematical modeling approaches (Chapter 1, section 1.1);
- Showing the connection between optimality conditions from the duality theory (2.6) and from the Karush-Kuhn-Tucker (2.8) points of view (Chapter 2, Observation 2.1);
- The full development of an Infeasible IPM and the detailed description of the algorithm (Chapter 3, Algorithm 3);
- In Chapter 4, obtaining the energy conservation equation (4.14) –which is part of the DC network model– from the AC transmission line power flow equations (4.3) and (4.4);
- Also in Chapter 4, obtaining the transportation model of the STEP problem (4.25) from the relaxation of the DC model (4.19) and a full description of its objective function and every constraint contained in the model;

- A detailed example of modeling of the STEP problem with and without redispatch (Chapter 4, subsection 4.5.1);
- The use of an IPM within the STEP problem as a solver in a Constructive Heuristic Algorithm and its detailed description (Chapter 5, Algorithm 4);
- Introducing a novel Power System Analysis Toolbox for the test systems operation condition verification (Appendix C).

The following two small contributions are results of this work as well:

1. International Poster
    - Becerril, C., Mota, R. and Badaoui, M. *Interior point algorithm as applied to the transmission network expansion planning problem.* SIAM Conference on optimization, San Diego, California, USA. May 19-22, 2014.
2. Conference paper
    - Becerril, C., Mota, R. and Badaoui, M. *Solution to the static transmission expansion planning by a primal-dual interior point method.* 7° Congreso Internacional de Ingeniería Electromecánica y de Sistemas, CIIES 2014.

1.6 Scope and General Methodology

The scope of this thesis is to handle a static and deterministic formulation of the network expansion, modeled basically to cope with a regulated environment problem. The mathematical model is from the cost minimization standpoint through the transportation model and the integer solution is reached using an heuristic.

The STEP problem presents a nonconvex mixed optimization formulation and some modifications to the model are necessary in order to have a more tractable one. So, once we have relaxed the STEP problem through the transportation model and then into a continuous LP, we have two tasks when trying to give an expansion plan: one related with the solution of the LP and the other one related to the integer nature of the problem.

Therefore, the general methodology of this work is as follows: First, we use an interior point method in order to solve in an efficient way the LP resulting of the relaxation; this will give us a continuous solution for the expansion. Then, we use the Garver heuristic to manage the integer part of the problem. If the expansion indicates no more additions, we stop. If not, we reconfigure the topology network and the model into a continuous LP once again. The IPM is used as a solver one more time and we obtain a new solution (continuous) for the Garver heuristic which will work as an integer solver. This is repeated until no more additions are needed.

1.7 Thesis Outline

The remainder of this thesis is organized in 5 more chapters as follows:

Chapter 2. This chapter is devoted to the review of the optimality conditions from the duality and from the KKT (Karush-Kuhn-Tucker) point of view. Finally, at the end of the chapter we show the connection between these conditions and the central path concept which is the fundamental stone of the IPM philosophy.

Chapter 3. In this chapter we show in great detail the development of a logarithmic barrier primal-dual IPM. This algorithm is based on the first part of the work developed at the Statistics and Operations Research Department of the Princeton University by Vanderbei & Shanno (the algorithm LOQO for nonconvex nonlinear programming).

Chapter 4. The fourth chapter describes the work with the AC transmission line power flow equations in order to figure out the DC model of the STEP. Then, we apply the convex and continuous relaxation to obtain the transportation model as a full LP. Finally, as an example we show in detail the development of the 6-bus Garver's test system modeling process for the STEP problem with and without redispatch.

Chapter 5. This chapter presents the solution of the STEP problem using the Garver's Constructive Heuristic Algorithm for the two test systems. We show the solution with and without redispatch for the Garver's system and the solution with redispatch for the 24-bus IEEE test system.

Chapter 6. The dissertation ends with the concluding remarks where we summarize the results and contributions of the thesis and some future work proposed.

The work contains four appendixes as well:

First, in **Appendix A** we show the *Fundamental Theorem of Linear Programming.* This is a very important theorem in LP because it ensures that if a LP has an optimum, this must be at a vertex of the feasible region. As this is related to the LP geometry and basic solutions, we give these definitions and terminology including simple theorems of convexity before arriving to the main theorem.

The algorithm described in chapter three is the first part of a general nonconvex nonlinear algorithm. Thus, the **Appendix B** provides the equivalence between this algorithm and a LP giving the corresponding formulas.

The power flow analysis is an essential tool for expansion planning process. We need to know how the system is operating in at least the steady state conditions before and after the expansion. Thus, in **Appendix C** we show all the power flow studies for each scenario for the 6-bus Garver's test system. For this purpose, we used the PSAT (Power

System Analysis Toolbox) which is an open source Matlab toolbox for electric power system analysis and simulation [58].

Finally, the data for the generation-demand and for the right of ways of the test systems studied in this work are given in **Appendix D**. The initial topology configuration of each system is given as well.

Chapter 2

Linear Program Optimality Conditions

The efforts of this chapter are focused on showing some theory of the Linear Programming problem, which is undoubtedly the optimization problem most frequently solved in practice. A review on the optimality conditions theory is given and we show that they can be obtained from duality theory and from the Karush-Kuhn-Tucker (KKT) conditions as well. Finally, the central path *concept is introduced as a consequence of these optimality conditions; this concept is an essential part of the interior point methods philosophy.*

2.1 Primal and Dual Linear Problems

A LP is a constrained optimization problem in which the objective function and each of the constrains are linear in the unknowns. Of course the set of constraints can include equality and/or inequality functions which defines the feasible solutions set (*feasible region*). However, for easy manipulation –which means adding or subtracting (nonnegative) slack variables in the inequality constraints– any LP can be transformed into the so-called *standard form*:

$$\begin{array}{rlcl} \text{minimize} & c^T x & & \\ \text{subject to} & A\,x & = & b \\ & x & \geq & 0 \end{array} \tag{2.1}$$

where $x, c^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The restriction $x \geq 0$ applies componentwise, that is, all components of the vector $x \in \mathbb{R}^n$ are required to be nonnegative.

The set

$$\mathcal{F}_p = \{x \in R^n \,|\, Ax = b, x \geq 0\} \tag{2.2}$$

is the feasible set for the *primal problem* (2.1) and it is called *primal feasible set.* A point $x \in \mathcal{F}_p$ is called a *feasible point*, and a feasible point x^* is called an *optimal solution* if $c^T x^* \leq c^T x \; \forall x \in \mathcal{F}_p$. If there is a sequence $\{x^k\}$ such that x^k is feasible and $c^T x^k \to -\infty$, then (2.1) is said to be *unbounded.*

The *dual problem* of (2.1) is defined as

$$\begin{array}{rlll} \text{maximize} & b^T y & & \\ \text{subject to} & A^T y & \leq & c \end{array} \tag{2.3}$$

or, in standar form

$$\begin{array}{rlll} \text{maximize} & b^T y & & \\ \text{subject to} & A^T y + s & = & c \\ & s & \geq & 0 \end{array} \tag{2.4}$$

where $y \in \mathbb{R}^m$ is the *dual* vector and $s \in \mathbb{R}^n$ is called *dual slack* vector. The set

$$\mathcal{F}_d = \{(y,s) \in R^m \times R^n \;|\; A^T y + s = c,\; s \geq 0,\; y \text{ free}\} \tag{2.5}$$

is the feasible set for the dual problem (2.4) and it is called *dual feasible set.*

According to this, the *primal-dual feasible set* is defined as follows

$$\mathcal{F} = \mathcal{F}_p \times \mathcal{F}_d = \{w = (x,y,s) \,|\, x \in \mathcal{F}_p,\; (y,s) \in \mathcal{F}_d\}$$

The pair (2.1)-(2.3) is called the *asymmetric form of duality.* The results showed in this chapter can be extended to the so-called symmetric form of duality which uses the canonical form in the primal.

2.2 Duality Theorems and Optimality

In the theory of duality there are some theorems which give an important relation between the primal and dual problems. For example, as it is stated below in the Lemma 2.1, a feasible solution to either of both problems (primal/dual) will produce a bound on the objective value of the other problem (dual/primal).

Lemma 2.1 (Weak duality lemma) *If $w \in \mathcal{F}$, then $c^T x \geq b^T y$.*

Proof. Since $w = (x,y,s) \in \mathcal{F}$ from dual $c^T = y^T A + s^T$ and from primal $Ax = b$, thus

$$c^T x = y^T Ax + s^T x = y^T b + s^T x = b^T y + s^T x \geq b^T y$$

where the inequality relation follows trivially from $s^T x \geq 0$. □

Corollary 2.1 *If $w^* = (x^*, y^*, s^*) \in \mathcal{F}$ is such that $c^T x^* = b^T y^*$, then w^* is the optimum.*

Proof. Let $w', w'' \in \mathcal{F}$, $w' = (x, y^*, s^*)$, $w'' = (x^*, y, s)$. From Lemma 2.1

$$c^T x \geq b^T y^* = c^T x^*$$

The primal (2.1) is a minimization problem, hence x^* is optimum for it. By the same reasoning from Lemma 2.1

$$b^T y \leq c^T x^* = b^T y^*$$

The dual (2.4) is a maximization problem, hence (y^*, s^*) is optimum for it. □

We call $c^T x - b^T y$ the *duality gap*, and is a measure of optimality. This was shown in the last corollary and it will be shown again but in a stronger result in the following theorem.

Theorem 2.1 (Strong Duality Theorem) *Let $\mathcal{F}$ be non-empty. Then, x^* is optimal for the primal* (2.1) *if and only if the following conditions hold:*

i. $x^* \in \mathcal{F}_p$;

ii. *there is* $(y^*, s^*) \in \mathcal{F}_d$;

iii. $c^T x^* = b^T y^*$.

Proof. See [59] Theorem 10.6, pp. 248. □

The *Strong Duality Theorem* tells that, whenever the primal problem has an optimal solution, the dual problem has one also and there is no duality gap. But what if the primal problem does not have an optimal solution? The *weak duality lemma* shows that the maximum cost of the dual is never above the minimum cost of the primal. Moreover, the optimal cost of the dual is always less than or –in the best case– equal to the optimal cost of the primal. Hence, if the cost of one of the problems is unbounded, then the other problem has no feasible solution. This is formalized in the following theorem.

Theorem 2.2 *If one of* (2.1) *or* (2.4) *is unbounded then the other has no feasible solution.*

Proof. Lets proceed by contradiction. First lets suppose that the objective value of the primal problem (2.1) is unbounded below (This mean that there must exist a sequence $\{\bar{x}^k\}_{k=0}^{\infty}$, $\bar{x}^k \in \mathbb{R}^n$, $\bar{x}^k \in \mathcal{F}_p$ for every $k \in \{1, 2, \ldots\}$, such that $c^T \bar{x}^k \to -\infty$). Now suppose that there exist $(y, s) \in \mathcal{F}_d$; *i.e.*, a point in the dual problem (2.4) that is feasible. From this and premultiplying by $(\bar{x}^k)^T$, we obtain

$$(\bar{x}^k)^T A^T y + (\bar{x}^k)^T s = (\bar{x}^k)^T c$$

On one hand, we have that $(\bar{x}^k)^T A^T y + (\bar{x}^k)^T s = (A\bar{x}^k)^T y + (\bar{x}^k)^T s$ but from the feasibility of the primal, the feasibility of the dual and from hypothesis we have $(A\bar{x}^k)^T = 0$, $s \geq 0$ and $(\bar{x}^k)^T \geq 0$, respectively. Thus

$$(\bar{x}^k)^T A^T y + (\bar{x}^k)^T s \geq 0$$

On the other hand, from hypothesis $c^T \bar{x}^k = (\bar{x}^k)^T c < 0$, giving a contradiction.

It is not difficult to complete the proof by assuming that the dual objective is unbounded above and making symmetric arguments. □

When there are feasible solutions to the primal and dual and combining the weak duality lemma and its corollary, we can observe that each problem is seeking to reach each other in such a way that when their cost are equal, both solutions are optimal. The next theorem is even a stronger result than that of the Strong Duality Theorem, the *LP Duality Theorem.*

Theorem 2.3 (LP Duality Theorem) *If primal* (2.1) *and dual* (2.4) *both have feasible solutions then both problems have optimal solutions and the optimal objective values of the objective functions are equal.*

Proof. See [52] Theorem 17.2, pp. 329. □

Optimality from Duality

From the LP Duality Theorem (Theorem 2.3), we have an easy way to verify whether or not a pair $x, (y, s)$ is optimal with the following system of linear inequalities and equations

$$\begin{array}{rcl} A^T y + s & = & c \\ Ax & = & b \\ x & \geq & 0 \\ s & \geq & 0 \\ c^T x - b^T y & = & 0 \end{array} \tag{2.6}$$

In the next section we will show that this conditions coincide with the Karush-Kuhn-Tucker optimality conditions for LP.

2.3 The Karush-Kuhn-Tucker Conditions

Kuhn and Tucker in 1961 developed the first order necessary optimality conditions for constrained nonlinear programming problems [60]. It was later discovered that W. Karush in his 1939 master's thesis at the University of Chicago had proven the same result [61].

First, lets consider the general nonlinear optimization problem for which $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^q$

$$\begin{array}{rrcl} \text{minimize} & f(x) & & \\ \text{subject to} & g(x) & = & 0 \\ & h(x) & \geq & 0 \end{array} \tag{2.7}$$

In order to state the KKT Theorem, the following definitions will be useful.

Definition 2.1 (Regular point, active constraint) *We say that x^* is a* regular point *for the constraints if the Jacobian matrix of the equality constraints and the gradient vectors of the* active inequality constraints *are linearly independent, i.e., if $\nabla g(x^*)$ and $\nabla h_j(x^*)\, \forall j$ with $h_j(x^*) = 0$ are linearly independent.*

Definition 2.2 (KKT point) *A feasible point x^* is called a* KKT point *if the following* KKT conditions *hold: There exist ($y^* \in \mathbb{R}^p$, $s^* \in \mathbb{R}^q$) such that (x^*, y^*, s^*) are satisfying*

i. $\nabla^T f(x^*) - \nabla^T g(x^*) y^* - \nabla^T h(x^*) s^* = 0$;

ii. $h^T(x^*)\, s^* = 0$;

iii. $s^* \geq 0$.

Here, y^* and s^* are called the *Lagrange* or *dual multipliers*.

Now, we are able to establish the KKT Theorem. Then, we will develop the KKT conditions for the standard form of a LP.

Theorem 2.4 (Karush-Kuhn-Tucker Theorem) *Let $f, g, h \in C^1$. Let x^* be a regular point and a local minimizer for the problem (2.7). Then x^* must be a KKT point.*

Proof. See [59] Theorem 5.25, pp. 125. □

If f is convex, g affine, and h is concave, then x^* is optimal if and only if x^* is a KKT point for (2.7). Thus, the necessary condition becomes sufficient.

LP Optimality Conditions

Now, lets consider the LP in standard form given in (2.1) that we rewrite here

$$\begin{array}{rrcl} \text{minimize} & c^T x & & \\ \text{subject to} & A\, x & = & b \\ & x & \geq & 0 \end{array}$$

where the vector $x \geq 0$ will play the role of the inequality constraints h in (2.7).

The convexity of the problem ensures that KKT conditions are sufficient for a global minimum. Hence, a KKT point for the LP given above –and thus by the Theorem 2.4 a global minimizer– is a point (x^*, y^*, s^*) which satisfies

$$\begin{array}{rcl} A^T y^* + s^* & = & c \\ Ax^* & = & b \\ x^* & \geq & 0 \\ s^* & \geq & 0 \\ (x^*)^T s^* & = & 0 \end{array} \tag{2.8}$$

The second and third expressions are because it is assumed that x^* is a feasible point.

The last condition $(x^*)^T s^* = 0$ is very important in interior point theory and is called *complementarity condition.*

Observation 2.1 *From the first condition* $s = c - A^T y$. *If we substitute this in the complementarity condition, we get* $x^T s = x^T c - x^T A^T y$, *but from the second expression we have that* $b^T = x^T A^T$, *thus*

$$x^T s = c^T x - b^T y$$

and we find that conditions (2.6) *and* (2.8) *are identical.*

It is not difficult to verify that the conditions (2.8) (or (2.6)) are sufficient for x^* to be a global solution of (2.1). Let $\hat{x}$ be any other feasible point, so that $A\hat{x} = b$ and $\hat{x} \geq 0$. Then, from the first expression of (2.8)

$$\begin{array}{rcl} c^T \hat{x} & = & (A^T y^* + s^*)^T \hat{x} \\ & = & (y^*)^T A\hat{x} + (s^*)^T \hat{x} \\ & = & (y^*)^T b + (s^*)^T \hat{x} \\ c^T \hat{x} & = & b^T y^* + \hat{x}^T s^* \end{array}$$

but $\hat{x}^T s^* \geq 0$ thus $b^T y^* + \hat{x}^T s^* \geq b^T y^*$ and since $b^T y^* = c^T x^*$ from (2.6), we finally have that $c^T \hat{x} \geq c^T x^*$. This means that no other feasible point can have a lower objective value than $c^T x^*$.

The complementarity conditions $x^T s = 0$ implies that at least one of the two variables x_j and s_j, $\forall j \in \{1, 2, \ldots, n\}$ has to be zero at the optimum. The way in which complementarity condition is dealt with, determines the type of optimization algorithms.

IPM perturb the complementarity condition and replace $x^T s = 0$ with $x^T s = \mu$, where the parameter μ is gradually reduced and eventually driven arbitrarily close to zero. The algorithm forces a reduction of μ and the partition of vectors x and s into zero and nonzero elements is gradually revealed as the algorithm progresses [62].

2.4 Analytic Center and the Central Path

The most appealing feature of IPM is that they show polynomial complexity. Nowadays, it is known that this is because they generate points near an infinitely smooth curve, called the *central path* [63]. Moreover, each element of the central path is an *analytic center* (for a proof we refer to [64]). Thus, in this section we will briefly discuss these very important concepts on IPM theory.

The analytic center

If we are at an interior point of a convex body represented by linear inequalities, we can take the steepest descent step in order to move into another point which results in a progress on the algorithm. However, this step does not make much advance unless the current point is *central*. This means being approximately equidistant from all of the bounds (Figure 2.1). The point which allows this nice centrality property is the so-called analytic center, and is the central point of an analytic measure of the convex body in IPM. The general idea behind this concept is the way in which the central-section algorithms iterates; you can see [55] and [65] for a review of these details.

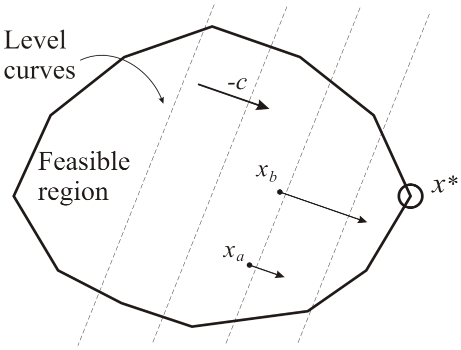

Figure 2.1: Centrality.

Central path

Definition 2.3 (Central Path) *The central path $\mathcal{C}$ is an arc of strictly feasible points in the primal-dual space, parametrized by a scalar $\mu > 0$, defined by the set of solutions (x_μ, y_μ, s_μ) of the system [66]:*

$$\begin{array}{rcl} A^T y + s & = & c \\ Ax & = & b \\ x & > & 0 \\ s & > & 0 \\ x^T s & = & \mu \end{array} \tag{2.9}$$

Observation 2.2 *The central path can be view as the set:* $\mathcal{C} = \{(x_\mu, y_\mu, s_\mu) | \mu > 0\}$, *satisfying* (2.9).

Observation 2.3 *The central path definition matches the optimality conditions except that*

1. *for the last expression called μ-complementarity condition. For which it is clear that if $\mu = 0$ we recover the dual complementarity condition of* (2.8) *and as $\mu \to 0$ the solution of* (2.9) *approaches the primal-dual solution; and*
2. *for the positivity conditions rather than nonnegativity. Regard this, every point of the central path are bounded away from zero allowing the centrality aforementioned and lying in the interior of the feasible region $\mathcal{F}$ of the primal and dual problems.*

Thus, the associated *central-path point* in primal-dual space is defined as the unique point that simultaneously satisfies the conditions of strict primal feasibility (second and third expressions), strict dual feasibility (first and fourth), and μ-complementarity (last equation) [67].

For each fixed μ, the points in $\mathcal{C}$ satisfying (2.9) can be viewed as sets of points in $\mathbb{R}^n$, $\mathbb{R}^p$, and $\mathbb{R}^q$, respectively. The corresponding points (analytic centers) when μ varies form the set of a trajectory called the central path (Figure 2.2).

One variant of the general primal-dual IPM is the *path following method* which restrict iterations to a neighborhood of the central path, avoiding points that are too close to the boundary (where $x = 0$ or $s = 0$) and where μ decreases so that the points can move every time closer to a KKT point. Here, instead of taking the pure Newton steps (which would be obtained in the case of being working with KKT conditions (2.8)), primal-dual IPM take Newton steps toward points on $\mathcal{C}$ (working with (2.9)).

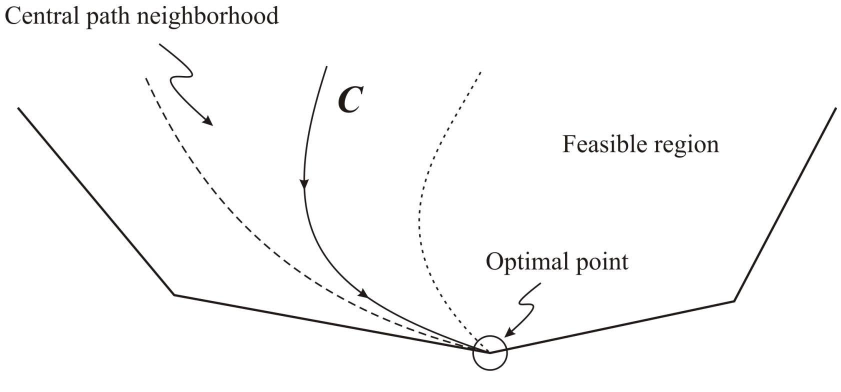

Figure 2.2: Central path [66].

Chapter 3

An Infeasible Interior Point Method

The aim of this chapter is to develop in detail an infeasible interior-point method*. The infeasible method that we will show here is a logarithmic barrier Primal-Dual Interior Point Method (PDIPM) and is based on the first part of the work of Vanderbei and Shanno for nonconvex and nonlinear problems [26]. In order to show the advantages of using the selected algorithm, we introduce the topic with the classic Fiacco and McCormick's method. The chapter begins with some general notions of the interior point algorithms and some comments on its relation with the transmission expansion problem.*

Given a general –continuous– optimization problem

$$\begin{array}{rrcl} \text{minimize} & f(x) & & \\ \text{subject to} & g(x) & = & 0 \\ & h(x) & \geq & 0 \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^q$, we will say that a point is *feasible* if it satisfies all the constraints, and *infeasible* otherwise. A point which satisfies the inequality constraints but may not satisfy the equality constraints is called *interior* [68].

When we use the term infeasible interior-point in a primal-dual method, the word "infeasible" refers to the fact that primal feasibility is not required by the algorithm at the beginning (and then enforced throughout) and it is only achieved as one approaches to an optimal solution. The modifier "interior-point" refers to the fact that the slack variables are required to be strictly positive at the beginning (and then throughout also) [69].

The first characteristic means that the power balance equations need not be satisfied at the initial point [31]. This feature can be particularly important when solving STEP problems in which we may deal often with disconnected buses, where power flow unsolvability is an issue; for instance, new generating or load buses or when an interconnection with a sub-system has to be done.

It is broadly accepted today that *infeasible-primal-dual* algorithms are the most efficient interior point methods. A number of attractive features of these methods follows from the fact that a logarithmic barrier method is applied to the primal and the dual problems at the same time [62]; this will be a good feature because of the nice properties of the logarithmic function as a barrier.

3.1 Barrier Methods

The first IPM were presented primarily in the form of barrier methods during the 1960's for solving nonlinear constrained problems. The barrier method is usually attributed to Frisch [70] but were Fiacco and McCormick who developed the mathematical theory of the subject in the context of nonlinear optimization [71]. Fiacco and McCormick noted the applicability of barrier methods to LP, however at that time it was the general perceived opinion that these methods would not be competitive with the simplex method and by the early 1980s barrier methods were almost without exception regarded as a closed chapter in the history of optimization. However, soon after Karmakar's publication, Gill et al. [72] showed a formal relationship between Karmakar's method and the classical logarithmic barrier method, marking the rebirth of the barrier methods.

In order to show the general idea behind barrier methods, lets consider the problem

$$\begin{array}{lrcl} \text{minimize} & f(x) & & \\ \text{subject to} & x & \geq & 0 \end{array}$$

The motivation of the barrier methods (and of the penalty methods, too) is to find an unconstrained minimizer of a composite function B –called *barrier function*– that reflects the original objective function $f(x)$ as well as the presence of the constraints. This can be achieved by combining the function f with a positively weighted "barrier" that prevents iterates from leaving the feasible region, named $\phi(x)$, and

$$B(x) = f(x) + \phi(x)$$

so that the problem is now written as

$$\text{minimize } B(x)$$

For example, one option for $\phi(x)$ could be:

$$\phi(x) = \begin{cases} 0 & \text{if} \quad x \text{ is feasible;} \\ +\infty & \text{if} \quad x \text{ is infeasible.} \end{cases}$$

However, this may result in a wildly discontinuous function which could be impossible to minimize. Thus, we need a well-behaved function that remains feasible but preserves nice properties (such as smoothness). This suggests the following desirables properties for $\phi(x)$ [73, 74]:

i. Continuity at all points of the interior of the feasible region

ii. Positivity, *i.e.*, $\phi(x) \geq 0$

iii. For any sequence of points in the interior converging to a point on the boundary of the feasible region, $\phi(x) \to +\infty$

A good choice for the last requirements is a logarithmic function, *i.e.*, we can define

$$\phi(x) = -log(x)$$

In fact, this is the overwhelmingly predominant barrier today perhaps for its connection with Karmakar's method, and was the basis of the Fiacco and McCormick's proposal.

3.1.1 The Fiacco and McCormick's logarithmic barrier method

The classical logarithmic barrier method of Fiacco and McCormick [71] was designed to solve the problem

$$\begin{array}{rll} \text{minimize} & f(x) & \\ \text{subject to} & h(x) & \geq \;\; 0 \end{array} \tag{3.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ are twice continuously differentiable in $\Omega = \{x \in \mathbb{R}^n \,|\, h(x) \geq 0\}$.

Fiacco and McCormick defined the barrier function

$$B(x, \mu^k) = f(x) - \mu^k \sum_{i=1}^{q} log(h_i(x))$$

where $\mu^k > 0$. It is important to see how $B(x, \mu^k)$ can retain the smoothness properties of $f(x)$ and $h(x)$ as long as $h(x) > 0$.

Now, instead of considering the original problem (3.1), the idea is to minimize the barrier function defined above. This is, to solve –for every index k– a sequence of unconstrained problems of the form

$$\min_x B(x, \mu^k) \tag{3.2}$$

with $\lim\limits_{k \to +\infty} \mu^k = 0$. The most important contribution of Fiacco & McCormick was to prove (see [71]), that under certain conditions for the functions $f(x)$ and $h(x)$ –that includes the regularity assumptions– and while $\mu^k \to 0$, the sequence $\{x(\mu^k)\}$ generated by (3.2) converges to x^*, which is a local minimizer for (3.1). The sequence $\{x(\mu^k)\}$ is known as the *barrier central path* [75]. Thus, we need to alternate between solving (3.2) and decreasing the value of μ for the next iteration.

The choise of the scheme for reducing μ^k is not an easy issue, however the simplest way to handle this decrease is by choosing a parameter $\gamma \in (0, 1)$ –usually 1/10– and set $\mu^{k+1} = \gamma\mu^k$ with μ^0 sufficiently large.

The Algorithm 1 shows the Fiacco & McCormick method for solving (3.1).

Algorithm 1: Fiacco & McCormick's Algorithm

```
Data: $\varepsilon > 0$, $\gamma \in (0, 1)$ and $\mu^0$ sufficiently large
Result: The optimum value $x^*$
begin
    $k \leftarrow 0$;
    Choose $x^k$ such that $h(x^k) > 0$;
    while $\mu^k > \varepsilon$ do
        Compute the unconstrained minimum $x(\mu^k) \leftarrow \min_x B(x, \mu^k)$;
        Set $\mu^{k+1} = \gamma\mu^k$;
        $k \leftarrow k + 1$;
    end
    $x^* \leftarrow x^k$;
end
```

Of course, since the logarithmic barrier function is always applied in the interior of the set defined by the inequality constraints, the need for the next assumption is clear: The set $\{x \,|\, h(x) > 0\}$ is non-empty.

This assumption clearly shows the need of feasibility which arises from the fact that minimizing (3.2) requires a feasible initial –and subsequent– estimate in order to avoid troubles with the domain of $B(x, \mu^k)$. In fact, IPM were developed under the assumption that the initial point is feasible and interior. However, for a general LP problem, computing a feasible point is as difficult as computing an optimal solution [68].

Another drawback of the classical logarithmic barrier method formulation arises from problems which do not include equality constraints. To solve this, Fiacco and McCormick added a penalty term to the barrier formulation to transform a more general problem like (2.7) in

$$\min_x F(x, \mu^k) = f(x) - \mu^k \sum_{i=1}^{q} log(h_i(x)) + \frac{1}{\mu^k} \sum_{i=1}^{p} (g_i(x))^2$$

Here a penalty term is added to assure that the equality constraints are driven to zero as $\mu \to 0$. However, this term can also be shown to be very ill-conditioned as $\mu \to 0$ [76].

The infeasible logarithmic barrier PDIPM proposed by Vanderbei and Shanno [26] does not need initial feasibility and was developed to handle equality constraints also.

3.2 An Infeasible IPM

Even though we will use the IPM for LP, we keep the general mathematical treatment given in [26] in order to let open the possibility of an extension. Thus, through this chapter we will consider the general optimization problem given in (2.7) which we recall here:

$$\begin{array}{rrcl} \text{minimize} & f(x) & & \\ \text{subject to} & g(x) & = & 0 \\ & h(x) & \geq & 0 \end{array} \tag{3.3}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^q$. f, g, $h \in C^2(\mathbb{R})$, *i.e.*, are twice continuously differentiable.

The Vanderbei and Shanno's algorithm follows the general idea of the logarithmic barrier PDIPM, which is to transform the problem (3.3) into a sequence of unconstrained problems [77]. We can perform this by following the next steps:

1. Add a vector s of slack variables to the set of inequality constraints $h(x)$ of (3.3) in order to transform them into another set of equality constraints; according to this, the original problem is written as:

$$\begin{array}{rrcl} \text{minimize} & f(x) & & \\ \text{subject to} & g(x) & = & 0 \\ & h(x) - s & = & 0 \\ & s & \geq & 0 \end{array} \tag{3.4}$$

   where x and s are the vectors of *primal variables*; $s \in \mathbb{R}^q_+$.

2. Use a barrier function to handle implicitly the remaining inequalities in (3.4) generated by the nonnegativity slack variables conditions. The classical logarithmic barrier function of Fiacco and McCormick were used in [26] to reformulate the problem (3.4) as follows:

$$\begin{array}{lrcl} \text{minimize} & f(x) - \mu^k \displaystyle\sum_{i=1}^{q} log(s_i) & & \\ \text{subject to} & g(x) & = & 0 \\ & h(x) - s & = & 0 \end{array} \tag{3.5}$$

   Now, because of the logarithmic term, $s \in \mathbb{R}^q_{++}$. Note that if the scalar μ (called *barrier parameter*) gets very close to 0, we obtain a good stand-in formulation for

(3.4). Here, for each μ we get an interior point, and as μ gets closer to zero this interior point moves closer to the optimal solution of the original optimization problem (in our case, LP).

Although the original problem could be a LP, by introducing the barrier term, every subproblem will be now a NLP.

3. Transform the latter subproblem into an unconstrained minimization problem. Note that if we fix μ^k, we can transform (3.5) into an unconstrained equivalent problem by using its related Lagrangian function.

   The Lagrangian for the subproblem (3.5) is:

$$\mathcal{L}_\mu(w;\mu^k) = f(x) - \mu^k \sum_{i=1}^{q} log(s_i) - \lambda^T g(x) - \pi^T \left[h(x) - s\right] \tag{3.6}$$

   where $(\lambda,\pi) \in \mathbb{R}^p \times \mathbb{R}^q$ are the *dual variables* of the problem, *i.e.* the Lagrange multipliers of the equality restrictions and inequality restrictions, respectively. Here, $w := (x, s, \lambda, \pi)$ is defined as the vector that contains all vector variables.

Again, in the same way as in the Fiacco and McCormick's method (see section 3.1.1), instead of solving (3.5) we will solve –for every μ^k– the following unconstrained minimization problem

$$\min \quad \mathcal{L}_\mu(w;\mu^k) \tag{3.7}$$

where $\mathcal{L}_\mu(w;\mu^k)$ is the Lagrangian given by (3.6).

3.2.1 Optimality conditions

A local minimizer for (3.7) is characterized by the KKT optimality conditions (see section 2.3), which in the case of unconstrained optimization problems, are reduced to the single requirement $\nabla f(x^*) = 0$ and it will be stated formally in the next theorem.

Theorem 3.1 *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}$ *is* $C^1(\mathbb{R}^n)$. *If* x^* *is a local minimum of* f *on* $\mathbb{R}^n$ *then* $\nabla f(x^*) = 0$.

Proof. Proceeding by contradiction, suppose that x^* is a local minimum but $\nabla f(x^*) \neq 0$. The idea is to show the existence of another point $z \in \mathbb{R}^n$, sufficiently close to x^* such that $f(z) < f(x^*)$.

Let x^* be the initial point and $z = x^* - \alpha \nabla f(x^*)$ a point in the direction of the vector $-\nabla f(x^*)$, scaled by $\alpha > 0$. By Taylor expansion around x^*:

$$f(z) = f(x^*) - \alpha \nabla f(x^*)^T \nabla f(x^*) + o(\alpha) = f(x^*) - \alpha ||\nabla f(x^*)||^2 + o(\alpha)$$

where $o : \mathbb{R} \to \mathbb{R}$ is such that $o(s)/s \to 0$ when $s \to 0$. Since $||\nabla f(x^*)|| \neq 0$, for a small enough $\alpha > 0$,

$$f(z) < f(x^*)$$

therefore, x^* could not be a local minimum, which is a contradiction. $\square$

Observation 3.1 *For $n = 1$, Theorem 3.1 reduces to the known statement: If $x^* \in \mathbb{R}$ is a local minimum then $f'(x^*) = 0$.*

KKT system construction and the central path for the problem

According to the Theorem 3.1, the first order necessary optimality conditions for the Lagrangian is obtained by taking $\nabla \mathcal{L}_\mu(w; \mu^k) = 0$. Thus, computing the partial derivatives for (3.6)

$$\begin{aligned} \nabla_x \mathcal{L} &= \nabla f(x) - \nabla g^T(x)\,\lambda - \nabla h^T(x)\,\pi = 0 \\ \nabla_s \mathcal{L} &= -\mu^k\, \nabla\left(\sum_{i=1}^{q} log(s_i) \right) + \nabla(\pi^T s) = 0 \end{aligned} \tag{3.8}$$

for this last equation

$$\nabla\left(\sum_{i=1}^{q} log(s_i) \right) = \left[\frac{1}{s_1}\ \frac{1}{s_2} \dots \frac{1}{s_q} \right]^T$$

Defining the diagonal matrix

$$S = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_q \end{bmatrix}$$

we get

$$S^{-1} = \begin{bmatrix} \frac{1}{s_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{s_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s_q} \end{bmatrix}$$

Using a vector of all ones $e \in \mathbb{R}^q$

$$S^{-1}e = \left[\frac{1}{s_1}\ \frac{1}{s_2} \cdots \frac{1}{s_q}\right]^T = \nabla\left(\sum_{i=1}^{q} log(s_i)\right)$$

On the other hand

$$\nabla(\pi^T s) = \nabla\left(\sum_{i=1}^{q} \pi_i s_i\right) = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_q \end{bmatrix} = \pi$$

If we remake the same process for vector π (in the same way than with the vector s), Π would be a diagonal matrix with elements π_i and we can write $\Pi\, e = \pi$.

These results yield

$$\nabla_s \mathcal{L} = -\mu^k S^{-1} e + \Pi\, e = 0$$

Finally, premultiplying this equation by S

$$\nabla_s \mathcal{L} = -\mu^k e + S\,\Pi\, e = 0 \tag{3.9}$$

The other two partial derivatives are

$$\nabla_\lambda \mathcal{L} = -g(x) = 0 \tag{3.10}$$

$$\nabla_\pi \mathcal{L} = -h(x) + s = 0 \tag{3.11}$$

From (3.8) to (3.11), the resulting *perturbed primal-dual system* is

$$\begin{array}{rcl} \nabla f(x) - \nabla g^T(x)\,\lambda - \nabla h^T(x)\,\pi & = & 0 \\ -\mu^k e + S\,\Pi\, e & = & 0 \\ -g(x) & = & 0 \\ -h(x) + s & = & 0 \end{array} \tag{3.12}$$

Observation 3.2 *The second equation of* (3.12) *can be written in the form*

$$S\,\Pi\, e = \mu^k e$$

or

$$s^T \pi = \mu^k \tag{3.13}$$

It is clear that the right side of (3.13) *is a strictly positive scalar due to the barrier parameter* μ^k. *This in fact implies the strict condition* $\pi \in \mathbb{R}^q_{++}$ *due to the presence of the* *logarithmic barrier which makes that the vector of primal slack variables s must be strictly feasible,* $s \in \mathbb{R}^q_{++}$*.*

Thus, the perturbed primal-dual system gives the central path for the problem, where the third and fourth equations of (3.12) *together with* $s \in \mathbb{R}^q_{++}$*, ensure primal (strict) feasibility (equivalent to the problem* (3.4)*). On the other hand, the first equation of* (3.12) *together with the implicit condition* $\pi \in \mathbb{R}^q_{++}$*, guaranties the dual (strict) feasibility.*

As it can be observed in this section, there are many subjects that they have to be solved. It will be a description of each of these subjects in the next sections.

3.3 Minimization of the Lagrangian

Numerical methods for *nonlinear unconstrained optimization problems* are iterative. At the $k-th$ iteration, a current approximate solution x^k is available. A new solution x^{k+1} is computed by certain techniques, and this process is repeated until a solution point can be accepted as an optimal solution. There are two fundamental iterative strategies to compute x^{k+1}: *trust region methods* and *line search methods*.

Trust region methods try to find the next solution point within a region called the trust region which is normally a set (say a ball or box) centered at the current iterate [78]. Byrd et al. proposed in [79] an interior point algorithm for solving large nonlinear programming problems which followed a barrier approach and incorporates trust region strategies.

Line search type methods for minimization search the next solution point in a line which follows a descent direction, and is the classical method for searching points in optimization algorithms ([66] and [80] are great references for a general description of this strategy). In the reference [26], they primarily used a line search algorithm in order to solve (3.7); we will show this in the next subsections.

In the Algorithm 2, we show a typical line search technique for an unconstrained optimization problem.

Recalling the line search paradigm for our problem, given a current solution point $w^k = (x, s, \lambda, \pi)$ the next step is to find a search direction $\Delta w^k = (\Delta x, \Delta s, \Delta\lambda, \Delta\pi)$ and a positive scalar α^k (called the step length), and then compute a new solution point w^{k+1} (in this case for every μ^k)

$$w^{k+1} = w^k + \alpha^k \Delta w^k \tag{3.14}$$

in such a way that with this new point a descent on the function that we are minimizing is obtained.

From (3.14), it is clear that two parameters for every iteration (the search direction and the step length) and one data for the initial computation (the initial point) are needed.

```
Algorithm 2: Descent Unconstrained Optimization Algorithm
Result: The optimum value $x^*$
begin
    $k \leftarrow 0$;
    Take a starting point $x^k$;
    while Termination criterion is not fulfilled do
        Determine a descent direction $\Delta x^k$;
        Compute a step length $\alpha^k$;
        Update the point: $x^{k+1} = x^k + \alpha^k \Delta x^k$;
        $k \leftarrow k + 1$;
    end
    $x^* \leftarrow x^k$;
end
```

3.3.1 Computing the search directions

It is well known that Newton's method is very efficient for linear and convex quadratic programming [26]. In order to obtain the search directions we will use the Newton's method. The directions can be obtained by getting only one step of the method to the system equations (3.12). This is why we will drop the index k (except for μ which indicates the current iterate value for it).

Considering the primal dual system (3.12) as a general system of equations $F(w) = 0$, the Newton's method forms a linear model for F around the current point and obtains the search direction Δw by solving:

$$J_{F(w)} \Delta w = -F(w) \tag{3.15}$$

where $J_{F(w)}$ stands for the Jacobian of $F(w)$ which in this case is the Hessian of the Lagrangian function (3.6), *i.e.*, $J_{F(w)} = \nabla^2 \mathcal{L}_\mu(w)$. For the Lagrangian given, we have

$$\nabla^2 \mathcal{L}_\mu(w) = \begin{bmatrix} \nabla^2 f(x) - \nabla^2 g^T(x)\,\lambda - \nabla^2 h^T(x)\,\pi & 0 & -\nabla g^T(x) & -\nabla h^T(x) \\ 0 & \Pi & 0 & S \\ -\nabla g(x) & 0 & 0 & 0 \\ -\nabla h(x) & I & 0 & 0 \end{bmatrix}$$

Thus we can write

$$\nabla^2 \mathcal{L}_\mu(w)\,\Delta w = \begin{bmatrix} -\nabla f(x) + \nabla g^T(x)\,\lambda + \nabla h^T(x)\,\pi \\ \mu^k e - S\,\Pi\,e \\ g(x) \\ h(x) - s \end{bmatrix} \tag{3.16}$$

Defining

$$\begin{aligned} H(x,\lambda,\pi) &= \nabla^2 f(x) - \nabla^2 g^T(x)\,\lambda - \nabla^2 h^T(x)\,\pi \\ D(x) &= -\nabla g(x) \\ E(x) &= -\nabla h(x) \end{aligned}$$

we can simplify equation (3.16) and write it as follows

$$\begin{bmatrix} H(x,\lambda,\pi) & 0 & D^T(x) & E^T(x) \\ 0 & \Pi & 0 & S \\ D(x) & 0 & 0 & 0 \\ E(x) & I & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta s \\ \Delta\lambda \\ \Delta\pi \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - D^T(x)\lambda - E^T(x)\pi \\ \mu^k e - S\,\Pi\, e \\ g(x) \\ h(x) - s \end{bmatrix} \tag{3.17}$$

The system (3.17) is not symmetric, but is easily symmetrized by multiplying the second equation by S^{-1}. Thus, the Newton's system for (3.12) is

$$\begin{bmatrix} H(x,\lambda,\pi) & 0 & D^T(x) & E^T(x) \\ 0 & S^{-1}\,\Pi & 0 & I \\ D(x) & 0 & 0 & 0 \\ E(x) & I & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta s \\ \Delta\lambda \\ \Delta\pi \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - D^T(x)\lambda - E^T(x)\pi \\ \mu^k S^{-1} e - \Pi\, e \\ g(x) \\ h(x) - s \end{bmatrix} \tag{3.18}$$

It should be noted that this is a Newton direction toward a point on the central path, *i.e.*, a point $(\Delta x_\mu, \Delta s_\mu, \Delta\lambda_\mu, \Delta\pi_\mu) \in \mathcal{C}$, called *the centering direction*. A pure Newton direction –called *the affine-scaling direction*– aims directly for a point at which the KKT conditions are satisfied and would be obtained when $\mu = 0$.

3.3.2 Computing the step lengths

In order to obtain the step length, we can compute a common value for both, primal and dual variables [31]. However, if the problem is a LP, it is recommended to compute a step length by a separated way, *i.e.*, one step length α_p^k for primal variables and another step length α_d^k for dual variables [26]; this is the simplest step length procedure in which the goal is to hold the strict positivity conditions $(s, \pi) \in \mathbb{R}^q_{++} \times \mathbb{R}^q_{++}$. Thus, we will develop the procedure to obtain the step length focused on this conditions.

Let $w^k := (x^k, s^k, \lambda^k, \pi^k)$ be the current point. Assume that this point satisfy the strict positivity conditions $(s^k, \pi^k) \in \mathbb{R}^q_{++} \times \mathbb{R}^q_{++}$. We must ensure that the new point remains positive, this is, that $s^{k+1} > 0$ and $\pi^{k+1} > 0$, *i.e.*, that $s^k + \alpha^k \Delta s^k > 0$ and $\pi^k + \alpha^k \Delta\pi^k > 0$. Working for s (the other is straightforward)

$$s^k > -\alpha^k \Delta s^k$$

In a line search paradigm $\alpha^k > 0$, and the feasibility of the current point guarantees that $s^k > 0$. Thus, we can write

$$\frac{1}{\alpha^k} > -\frac{\Delta s^k}{s^k} \tag{3.19}$$

It is easy to see that even if we choose

$$\frac{1}{\alpha^k} = \max\left\{-\frac{\Delta s^k}{s^k}\right\}$$

the strict inequality in (3.19) is not always ensured, because at least for one value of the quotient $\Delta s^k/s^k$, both sides will be equal. To avoid this, we can take $t > 1$ and set

$$\frac{1}{\alpha^k} = t\ \max\left\{-\frac{\Delta s^k}{s^k}\right\}$$

If $\alpha_0 = t^{-1}$ then $\alpha_0 \in (0,1)$ and the last expression can be written as

$$\alpha^k = \alpha_0 \left(\max\left\{-\frac{\Delta s^k}{s^k}\right\}\right)^{-1}$$

Thus, the maximum step length that we can compute and obtain for each variable would be given by

$$\begin{aligned} \alpha_p^k &= \alpha_0 \left(\max\left\{-\frac{\Delta s^k}{s^k}\right\}\right)^{-1} \\ \alpha_d^k &= \alpha_0 \left(\max\left\{-\frac{\Delta \pi^k}{\pi^k}\right\}\right)^{-1} \end{aligned} \tag{3.20}$$

A value of $\alpha_0 = 0.95$ is recommended in practice [26, 81].

Variables updating

The new values for the primal and the dual variables are calculated as follows

$$\begin{aligned} x^{k+1} &= x^k + \alpha_p^k \, \Delta x^k \\ s^{k+1} &= s^k + \alpha_p^k \, \Delta s^k \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \lambda^{k+1} &= \lambda^k + \alpha_d^k \, \Delta \lambda^k \\ \pi^{k+1} &= \pi^k + \alpha_d^k \, \Delta \pi^k \end{aligned} \tag{3.22}$$

3.4 Reducing the Barrier Parameter

According to the Fiacco and McCormick's Theorem, one of the main issues in this kind of IP methods is to establish how to decrease μ.

As it was aforementioned, the simplest way to reduce the barrier parameter is decreasing its value by a fixed factor up to a given lower bound, for example $\mu^{k+1} = \frac{1}{10}\mu^k$, or as Monteiro et al. suggested [82, 83]

$$\mu^{k+1} = \mu^k \left(1 - \frac{0.1}{\sqrt{n}}\right)$$

However, the experience has shown that μ should not be decreased too fast –as in the first case– because this may result in non-convergence or with a very small multiple as in the Monteiro's formula because they are hopelessly slow in practice.

3.4.1 Duality gap criterion

In LP problems, the decrease of the value of μ is usually estimated based on the predicted decrease of the duality gap

$$\mu^{k+1} = \frac{c^T x - b^T y}{\phi(n)} \tag{3.23}$$

where

$$\phi(n) = \left\{ \begin{array}{rl} n^2, & n \leq 5\,000 \\ n\sqrt{n}, & n > 5\,000 \end{array} \right.$$

However, (3.23) is positive *only* for feasible primal and dual values because the weak duality lemma, where $c^T x \geq b^T y$ (Chapter 2). Lusting et al. did not overlooked this fact and proposed some modifications for this [82].

Instead of the latter, we can use the complementarity gap as it is shown next.

3.4.2 Complementarity gap criterion

If the iterates converge to an optimum, then the sequence generated by the value of the complementarity conditions $(s^k)^T \pi^k$ must converge to zero. This suggests that μ^k could be reduced based on a predicted decrease of the *complementarity gap* given by

$$\rho^k = (s^k)^T \pi^k \tag{3.24}$$

Thus we can write

$$\mu^{k+1} = \frac{\rho^k}{q} \tag{3.25}$$

This is a better criterion to get a reduction for the value of μ, however this follows the centering direction (see section 2.4 and subsection 3.3.1) and is biased strongly toward the interior of a nonnegative orthant where s and π are strictly positive and makes little progress in reducing μ. Then, we can accelerate the reduction by doing

$$\mu^{k+1} = \sigma \frac{\rho^k}{q} \tag{3.26}$$

where $\sigma \in (0, 1)$, called *the centering parameter*, is looking for a perfect mixture between improving centrality and reducing μ. Note that if $\sigma \to 1$ we would approach to the centering direction and as $\sigma \to 0$ we would approach to the affine-scaling direction.

We can choose a fixed value of $\sigma = 0.1$ as in this work, or as is proposed in [31], dynamically chosen as follows

$$\sigma^{k+1} = max\{0.99\sigma^k,\, 0.1\}$$

with $\sigma^0 = 0.2$.

3.4.3 Vanderbei and Shanno's criterion

Based on their experience, Vanderbei and Shanno [26] pointed that this infeasible algorithm performs the best when the complementarity products $s^T\pi$ approach zero at a uniform rate and used

$$\xi = \frac{min_i\, s_i\pi_i}{\pi^T s/q}$$

in order to measure the distance from uniformity.

Note that ξ is in fact a measure of centrality of the current point. Thus, their proposal is based on the theoretical IPM results that when the trajectory is far from centrality, a larger μ promotes centrality for the next iteration, whereas when the trajectory is close to the central path, a small μ should be chosen. Following these ideas, they suggested an heuristic for the choice of μ given by

$$\mu^{k+1} = \zeta\, min\left((1-\alpha_0)\frac{1-\xi}{\xi},\, 2 \right)^3 \frac{s^T\pi}{q} \tag{3.27}$$

where $\alpha_0 \in (0, 1)$ is the step length parameter described in subsection 3.3.2, which in [26] defaults to 0.95, and ζ is a settable scale factor, which defaults to 0.1.

3.5 The Stopping Criteria

Ideally speaking, it is expected that optimality conditions (3.12) are satisfied. This means that, in order to have an approximate local minimum, we need that primal, dual and complementarity conditions are fulfilled in some sense. Thus, the following stopping criteria must be satisfied:

1. **Primal stopping criterion.** In the case of the *primal conditions*, it is expected that

   $$-g(x^k) = 0$$

   and

   $$-h(x^k) + s = 0$$

   However, we can be content if the maximum norm between all the components of each vector fall below some tolerance.

   Thus, we can use the l_∞ norm defined for an n-dimensional vector x as

   $$\|x\|_\infty = max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

   in order to set up that

   $$\|g(x^k)\|_\infty \leq \varepsilon_1$$

   But, for the second condition –since $h(x^k) \geq 0$– it is enough to write

   $$max\{-h(x^k)\} \leq \varepsilon_1$$

   Finally, we can combine both expressions and obtain the *primal stopping criterion* given by [36, 38]

   $$v_1^k = max\left\{\|g(x^k)\|_\infty,\, max\{-h(x^k)\}\right\} \leq \varepsilon_1 \tag{3.28}$$

2. **Dual stopping criterion.** In the case of the *dual conditions*, it is expected that

   $$\nabla f(x) - \nabla g^T(x)\,\lambda - \nabla h^T(x)\,\pi = 0$$

   Again, we can be content if the maximum norm between all the components of the last vector fall below some tolerance. Thus, we expect that

   $$\|\nabla f(x) - \nabla g^T(x)\,\lambda - \nabla h^T(x)\,\pi\|_\infty \leq \varepsilon_1$$

   Moreover, we can have a better approximation to the solution if we divide (scale) the length (given by the norm) by a small number, say for example by the l_2 norm of the vector x given by

   $$\|x^k\|_2 = \sqrt{x^T x}$$

   and, in order to avoid the possibility of an undefined expression, we can write the *dual stopping criterion* as [36, 38]

   $$v_2^k = \frac{\|\nabla f(x^k) - \nabla g^T(x^k)\,\lambda^k - \nabla h^T(x^k)\,\pi^k\,\|_\infty}{1 + \|x^k\|_2} \leq \varepsilon_1 \tag{3.29}$$

3. **Complementarity stopping criterion.** According to the last discussions, we can use the complementarity gap (3.24) and write the *complementarity stopping criterion* as follows [36, 38]

$$v_3^k = \frac{\rho^k}{1 + \|x^k\|_2} \leq \varepsilon_2 \tag{3.30}$$

4. μ **stopping criterion.** Finally, we should consider the value of μ when it gets too close to zero, *i.e.*, the last stopping criterion is given by [31]

$$\mu^k \leq \varepsilon_\mu \tag{3.31}$$

Thus, if the criteria $v_1^k \leq \varepsilon_1$, $v_2^k \leq \varepsilon_1$, $v_3^k \leq \varepsilon_2$ and $\mu^k \leq \varepsilon_\mu$ are fulfilled, then the primal feasibility, (scaled) dual feasibility, (scaled) complementarity gap and the μ limit respectively are satisfied; moreover, when conditions (3.28), (3.29) and (3.30) are satisfied, the current iterate is a KKT point of accuracy ε_1 and ε_2.

Typical convergence tolerance values are $\varepsilon_1 = 10^{-4}$, $\varepsilon_2 = 10^{-6}$ and $\varepsilon_\mu \leq 10^{-12}$ [31, 38].

3.6 Initial Point

For this kind of methods, a feasible initial point is not mandatory. However, in order to define the logarithmic barrier terms and to avoid spurious solutions (points that satisfy the KKT equations but violate the positivity conditions), the strict positivity conditions on the primal slacks ($s \in \mathbb{R}^q_{++}$) and dual slacks ($\pi \in \mathbb{R}^q_{++}$) must be satisfied by the initial point and all subsequent iterates. Towards this end, the method iterates start from a point that meets these conditions and hold them by following a trajectory in the positive orthant of the space defined by the primal slack and dual slack variables.

Some techniques can be given for starting points. In [31] four approaches for estimate w^0 are given for the optimal power flow problem and some similar techniques could be useful for STEP problem. However, in this work we only accomplish the requirement of strict starting positivity condition on (s, π) by setting $w^0 = [1, 1, \ldots, 1]$.

3.7 The Infeasible PD-IPM Algorithm

The Algorithm 3 shows the infeasible primal-dual interior point method described in this chapter which will be used as a LP solver within the STEP problem process.

Algorithm 3: Infeasible Primal-Dual Interior-Point Algorithm

```
Data: $\varepsilon_1, \varepsilon_2$ and $\varepsilon_\mu$, $\sigma \in (0,1)$ and $w^0$ such that $(s, \pi) > 0$
Result: The optimal vector $w^*$
begin
    $k \leftarrow 0$;
    Compute the initial barrier parameter $\mu^k$ by (3.26) or (3.27);
    Form the initial perturbed primal-dual system (3.12);
    while (3.28) – (3.31) are not verified do
        Compute the Newton direction $\Delta w^k$ by (3.17) or by the reduced system
          (3.18);
        Obtain the primal $\alpha_p^k$ and dual $\alpha_d^k$ step length in the direction $\Delta w^k$ using
          (3.20);
        Update the primal $(x^k, s^k)$ and dual $(\lambda^k, \pi^k)$ variables by (3.21) and (3.22);
        Reduce the barrier parameter $\mu^k$ by (3.26) or (3.27);
        Update the perturbed primal-dual system (3.12);
        $k \leftarrow k + 1$;
    end
    $w^* \leftarrow w^k$;
end
```

Chapter 4

Problem Modeling and Relaxation

This chapter gives the development of the transportation model for the STEP problem. First we will construct in detail the DC STEP model which is a mixed-integer nonlinear *optimization problem. Then, we will relax the problem to a mixed-integer* linear *optimization problem to obtain the transportation model and we will match this model to the structure of* (1.1). *Finally, the model development for the Garver's system is showed, making way to the formulation* (3.3) *and* (3.4).

4.1 Transmission Line Power Flow

In steady state power system analysis, the lumped parameter $\pi-$equivalent model (Figure 4.1) is often used to model an overhead AC $l-th$ transmission line (linking a bus i with another bus k) and is characterized by a *series impedance* (z_l) and in each ending of the line by a *shunt admittance* (y_{l_0}).

Here, the series impedance of the line can be written as

$$z_l = r_l + jx_l$$

where r_l and x_l are referred to as the *resistance* and *reactance* of the line, respectively (the reactance is capacitive if $x_l < 0$ or inductive if $x_l > 0$).

Another important element in power system analysis and network equations is the *series admittance* of the line, which is defined as the reciprocal of the series impedance:

$$y_l = \frac{1}{z_l} = \frac{1}{r_l + jx_l} = \frac{r_l - jx_l}{r_l^2 + x_l^2} = g_l + jb_l \tag{4.1}$$

where g_l and b_l are referred to as the *conductance* and *susceptance* of the line, respectively; in actual transmission lines $r_l > 0$ and $x_l > 0$ thus g_l is positive whereas b_l is negative.

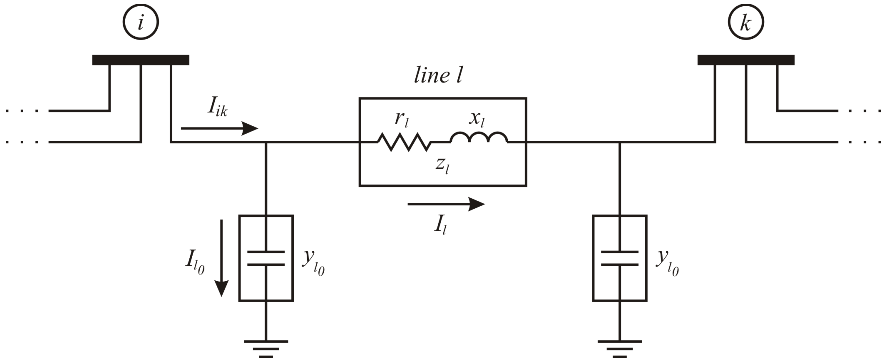

Figure 4.1: Pi configuration of a single transmission line.

The shunt admittance is in general a *shunt capacitance* (a parallel connection between the device and the electrical ground) which is an effect due to the electric fields between conductors. In many cases the value of g_{l_0} is too small so it could be neglected.

4.1.1 AC transmission line power flow

The current (I_{ik}) that will be sent from bus i to bus k through a $l\text{–}th$ transmission line is divided into two components (Figure 4.1): one component (I_l) flowing through the series impedance and another component (I_{l_0}) flowing through the shunt admittance. That is

$$I_{ik} = I_l + I_{l_0} = (V_i - V_k)y_l + V_i\, y_{l_0}$$

Analogously, the sending complex power (S_{ik}) through the $l-th$ transmission line is

$$\begin{aligned} S_{ik} &= V_i I_{ik}^* \\ &= V_i[(V_i - V_k)y_l + V_i y_{l_0}]^* \\ S_{ik} &= V_i[(V_i - V_k)y_l]^* + V_i[V_i y_{l_0}]^* \end{aligned}$$

Taking the exponential form for voltages and rectangular form for the admittances, and neglecting the conductance of y_{l_0}

$$\begin{aligned} V_i &= |V_i|e^{j\theta_i} & V_k &= |V_k|e^{j\theta_k} \\ y_l &= g_l + jb_l & y_{l_0} &= jb_{l_0} \end{aligned}$$

S_{ik} becomes

$$\begin{aligned} S_{ik} &= |V_i|e^{j\theta_i}\Big[(|V_i|e^{j\theta_i} - |V_k|e^{j\theta_k})(g_l + jb_l)\Big]^* + |V_i|e^{j\theta_i}\Big[j|V_i|e^{j\theta_i}b_{l_0}\Big]^* \\ &= \Big[|V_i|^2 - |V_i||V_k|e^{j(\theta_i-\theta_k)}\Big]\,(g_l - jb_l) - j|V_i|^2 b_{l_0} \\ S_{ik} &= \Big[|V_i|^2 - |V_i||V_k|\,\cos(\theta_i - \theta_k) - j|V_i||V_k|\,\sin(\theta_i - \theta_k)\Big](g_l - jb_l) - j|V_i|^2 b_{l_0} \end{aligned}$$

Then the complex power flow through a $l-th$ transmission line that links a bus i to a bus k is

$$\begin{aligned} S_{ik} &= g_l|V_i|^2 - g_l|V_i||V_k| \cos(\theta_i - \theta_k) - b_l|V_i||V_k| \sin(\theta_i - \theta_k) \\ &\quad -j\Big[g_l|V_i||V_k| \sin(\theta_i - \theta_k) + b_l|V_i|^2 - b_l|V_i||V_k| \cos(\theta_i - \theta_k) + |V_i|^2 b_{l_0}\Big] \end{aligned} \tag{4.2}$$

Thus, from (4.2) the real and reactive power flow through the $l-th$ transmission line is, respectively

$$P_{ik} = g_l|V_i|^2 - |V_i||V_k|\big(g_l \cos(\theta_i - \theta_k) + b_l \sin(\theta_i - \theta_k)\big) \tag{4.3}$$

and

$$Q_{ik} = -|V_i||V_k|\big(g_l \sin(\theta_i - \theta_k) - b_l \cos(\theta_i - \theta_k)\big) - |V_i|^2(b_l + b_{l_0}) \tag{4.4}$$

4.1.2 DC transmission line power flow

The DC transmission line power flow equations form an equivalent model that provides an approximate solution for a network carrying AC power, supplying all the necessary information in the context of planning. This model captures the physics of an AC transmission line power flow but in a relaxed form.

The DC transmission line power flow equations are obtained by making the following assumptions [84]:

Assumption A. The reactance in a transmission line is much bigger than its resistance ($x_l >> r_l$).

In general, the quotient $\frac{r_l}{x_l}$ is not high in transmission systems. In fact, high $\frac{r_l}{x_l}$ ratios are anomalous situations which have been investigated because they are related with convergence difficulties of the power flow algorithms and ill-conditioned power systems [85, 86, 87, 88, 89, 90]. Thus, we can consider $r_l \approx 0$. Indeed, from (4.1) we have $g_l \approx 0$. This leads to a first reduction for the couple of equations (4.3) and (4.4) as follows

$$P_{ik} = -|V_i||V_k|b_l \sin\theta_{ik} \tag{4.5}$$

$$Q_{ik} = |V_i||V_k|(b_l \cos\theta_{ik}) + |V_i|^2(b_{l_0} - b_l) \tag{4.6}$$

where $\theta_{ik} = \theta_i - \theta_k$.

Assumption B. The difference in the angle of the voltages phasors are small.

For stability reasons, in power systems one of the main issues is keeping angular separation between two buses as close as possible, and it is extremely rare to observe differences that exceeds an angular displacement of 30 to 35° across the line [91, 92]. Moreover, for normal operating conditions this angle separation is always less than 15°; we refer to the table C.1 of the Appendix C as an example, where $\theta_{ik} < 10°$.

Thus, it is possible to assume $\theta_{ik} \approx 0$ and as a consequence $\sin\theta_{ik} \approx \theta_{ik}$ and $\cos\theta_{ik} \approx 1$. This leads to a second reduction yielding

$$P_{ik} = -|V_i||V_k|b_l\theta_{ik} \tag{4.7}$$

$$Q_{ik} = |V_i||V_k|\, b_l + |V_i|^2(b_{l_0} - b_l) \tag{4.8}$$

Assumption C. The bus voltage magnitudes $|V_i|$ and $|V_k|$ are very close to 1.0 *p.u.*

In normal operating conditions the bus voltage magnitudes should fluctuate between 0.95 *p.u.* y 1.05 *p.u.* [91, 92]; we refer again to the table C.1 of the Appendix C as an example. Thus, we can assume that $|V_i| \approx 1$ and $|V_k| \approx 1$. This leads to a third reduction that yields

$$P_{ik} = -b_l\theta_{ik} \tag{4.9}$$

$$Q_{ik} = b_{l_0} \tag{4.10}$$

Finally, in short lines the shunt element b_{l_0} is neglected and in medium and long lines $b_l >> b_{l_0}$, thus $P_{ik} >> Q_{ik}$. As a consequence, the DC power flow equations consider only (4.9), *i.e.*, considering assumption A and (4.1), (4.9) is

$$P_{ik} = \frac{1}{x_l}\theta_{ik} \tag{4.11}$$

Observation 4.1 *It should be noted that equation* (4.11) *is an expression of Ohm's law for the equivalent DC network and so Kirchhoff's Voltage Law (KVL) is implicitly taken into account.*

4.2 DC Network Representation for the STEP Problem

When the DC network representation is used, we have to ensure that both Kirchhoff's laws are satisfied by two equivalent expressions. In this section, *power conservation* equations in each node (KCL equivalent) and *energy conservation* equations (KVL equivalent) are established.

4.2.1 Power conservation (The power balance equations)

The total real power injected by the $i-th$ bus (P_i) to the network is defined as the difference between the real power generation (P_{G_i}) and the real power demanding (P_{D_i}) (Figure 4.2), *i.e.*

$$P_i = P_{G_i} - P_{D_i} = \sum_{k\in\mathcal{K}} P_{ik}$$

where in STEP problem, $\mathcal{K}$ denotes the set of all buses connected to the bus i by an existent or a candidate transmission circuit.

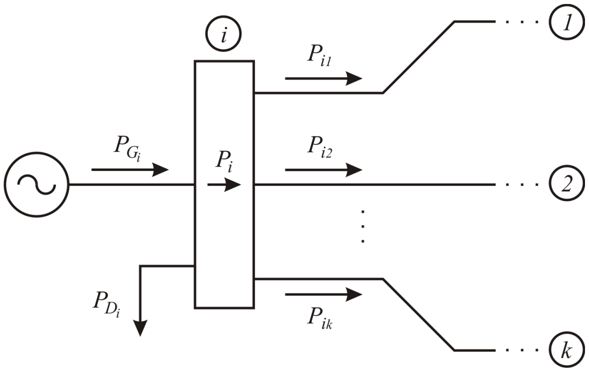

Figure 4.2: Power injected to the network.

Thus, the power conservation equation for each bus i is

$$-\sum_{k\in\mathcal{K}} P_{ik} + P_{G_i} = P_{D_i} \tag{4.12}$$

where we have to take into account that $P_{ki} = -P_{ik}$ since we are considering a lossless network.

Equation (4.12) gives an equivalent form of the KCL for the DC model.

Compact writing of the power balance equations

We can use the *node-branch incidence matrix* ($\mathcal{S}$) in order to obtain a reduced form of the power balance equations. The incidence matrix formulation allows to take implicitly into account that $P_{ki} = -P_{ik}$ and for the STEP problem has the following characteristics for its construction[1]:

1. Rows correspond to buses and columns to Right Of Ways (ROW).
2. The column for ROW (i, k) has exactly two non-zero entries: -1 in row i and $+1$ in row j.

Thus, equation (4.12) can be written as follows

$$\mathcal{S}f + g = d \tag{4.13}$$

---

[1] A *right of way* is a portion of land that is owned utility and could be available for the construction of new transmission circuits

where now in STEP problem, f is the vector with elements f_{ik} which will correspond to the real power flow P_{ik} through an existent transmission circuit in ROW (i, k) or even "through" a candidate for construction transmission circuit in ROW (i, k). Analogously, g is the vector with elements g_i which will correspond to the real power generation at the bus i, P_{G_i}, and d is the vector with elements d_i which will correspond to the real power demand at the bus i, P_{D_i}.

4.2.2 Energy conservation

It has been mentioned in *Observation 4.1* that (4.11) is an expression of Ohm's law for the equivalent DC network and so Kirchhoff's Voltage Law (KVL) is implicitly taken into account (energy conservation). Using the new notation, we can write

$$f_{ik} - \gamma_{ik}\theta_{ik} = 0$$

where again for STEP problem, γ_{ik} is the susceptance of a transmission circuit that could be a candidate (n_{ik}) or an already existent (n^0_{ik}). This dependency leads to an expression for the energy conservation for each bus i $(\forall k \in \mathcal{K})$ as follows [21]:

$$f_{ik} - \gamma_{ik}(n^0_{ik} + n_{ik})\theta_{ik} = 0 \tag{4.14}$$

It should be noted that equation (4.14) is nonlinear because of the products between the n_{ik} variables and the θ_{ik} variables.

4.3 DC Model for the STEP Problem

We are now able to write the STEP problem formulation using the DC (network) model.

4.3.1 Objective function

In this work, the objective of the STEP problem is to *minimize the construction cost*, represented as a linear function where the cost is directly proportional to the number of transmission circuits which will be built in a ROW. This can be written as:

$$Min \quad c^T n \tag{4.15}$$

where c^T is the transpose vector containing c_{ik} elements which correspond to the cost of construction of each transmission circuit in a ROW (i, k) and n is a vector of the n_{ik} variables representing the candidate transmission circuits in the ROW (i, k).

4.3.2 Constraints

DC network representation

Here, equations (4.13) and (4.14) are considered.

Power transfer limits

Due to security reasons, we must limit the transportation of power flows in the candidate and existent transmission circuits by a maximum power flow capacity per transmission circuit ($\overline{f}_{ik}$) in a ROW (i, k), which is an element of the vector $\overline{f}$. This can be considered by the following expression [21]:

$$|f| \leq (n^0 + n)\,\overline{f} \tag{4.16}$$

Here, n^0 is the vector containing existing transmission circuits elements n^0_{ik} in a ROW (i, k).

General constraints

Here, we set general constraints where the vector of maximum generator power output $\overline{g}$ and the vector which takes into account the space limitation for transmission circuit construction $\overline{n}$ are considered. This is written as follows

$$0 \leq g \leq \overline{g} \tag{4.17}$$

$$0 \leq n \leq \overline{n} \tag{4.18}$$

The vector $\overline{g}$ contains elements $\overline{g}_i$ which correspond to the maximum limit generator power output for the generation bus i. The vector $\overline{n}$ has $\overline{n}_{ik}$ as elements which refer to the maximum transmission circuits allowed per ROW.

The model

Finally, the STEP DC model is [21, 93]:

$$minimize \quad v = c^T n$$

$$subject\ to: \qquad \begin{gathered} \mathcal{S}f + g = d \\ f_{ik} - \gamma_k(n^0_{ik} + n_{ik})\theta_{ik} = 0 \\ |f| \leq (n^0 + n)\overline{f} \\ \underline{g} \leq g \leq \overline{g} \\ 0 \leq n \leq \overline{n} \\ n \in \mathbb{Z}^r,\ f \in \mathbb{R}^n \end{gathered} \tag{4.19}$$

The model given in (4.19) is a difficult mixed-integer, nonlinear, nonconvex optimization problem; mixed because of the integer variables of transmission circuit additions and nonlinear (and nonconvex) because of energy conservation equations.

4.4 Problem Relaxation

As it was introduced in the first chapter, the general idea of relaxation is to formulate a related problem which underestimates their objective function and/or their constraints. Of course the optimum value of the relaxed problem will be just a lower bound for that of the original problem, however this is a very useful approach when the problem is difficult to solve as it is the case in the DC model (4.19). This model can be relaxed in two possible forms: convex and continuous relaxation.

Lets begin formalizing the general idea of relaxation with a definition [59]:

Definition 4.1 (Relaxation) . *Given the problem to find,*

$$\begin{aligned} f^* &:= \inf_x \ f(x) \\ s.t. &\qquad x \in \mathcal{Q} \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ *is a given function and* $\mathcal{Q} \subseteq \mathbb{R}^n$*, we define a* relaxation *to the latter formulation to be a problem of the following form: find*

$$\begin{aligned} f_R^* &:= \inf_x \ f_R(x) \\ s.t. &\qquad x \in \mathcal{Q}_R \end{aligned}$$

where $f_R : \mathbb{R}^n \to \mathbb{R}$ *is a function with the property that* $f_R \leq f$ *on* $\mathcal{Q}$*, and where* $\mathcal{Q} \subseteq \mathcal{Q}_R$*.*

For example, by proper addition of slack variables, each constraint in (4.19) which is a mixed-integer, nonlinear, nonconvex problem, can be written as equalities and the feasible set for the model is

$$\mathcal{Q} = \left\{ x \in \mathbb{Z}^r \times \mathbb{R}^n \ : \ \begin{array}{rcl} h_N(x) & = & 0 \\ h_L(x) & = & 0 \end{array} \right\}$$

where h_N is the set of nonlinear and nonconvex constraints which contains energy conservation equations (4.14) and h_L is the set of linear constraints containing all the remaining constraints on (4.19).

By eliminating the h_N equations, we can obtain the *convex relaxed problem* and

$$\mathcal{Q}_R = \left\{ x \in \mathbb{Z}^r \times \mathbb{R}^n \ : \ h_L(x) \ = \ 0 \ \right\}$$

The *continuous relaxation* of the problem can be easily obtained by dropping the integer constraints, *i.e.*

$$\mathcal{Q}'_R = \left\{ x \in \mathbb{R}^n \ : \ h_L(x) \ = \ 0 \ \right\}$$

We will use these concepts to obtain a convex-continuous relaxed formulation of the DC model called *transportation model*.

4.5 Transportation Model for the Test Systems

The transportation model is the classical convex relaxation of the STEP DC model and it was first suggested by Garver in his seminal work [40].

As it was aforementioned, the difficulties of the DC model are associated with nonlinear constraints (4.14) in the network modeling. Thus –as in the previous section– we can define a relaxation on the feasible set by eliminating those nonlinear constraints, and the model becomes [20, 22, 93]:

$$min \quad v = c^T n \tag{4.20}$$

subject to :

$$\mathcal{S} f + g = d \tag{4.21}$$

$$|f| \leq (n^0 + n)\overline{f} \tag{4.22}$$

$$0 \leq g \leq \overline{g} \tag{4.23}$$

$$0 \leq n \leq \overline{n} \tag{4.24}$$

$$n \in \mathbb{Z}^r, \, f \in \mathbb{R}^n$$

In order to have the structure given by (1.1), lets work with the inequality constraints as follows:

First, inequality (4.22) can be written as

$$-(n^0 + n)\overline{f} \leq f \leq (n^0 + n)\overline{f}$$

i.e., as a couple of inequalities in the form

$$\begin{array}{rcl} f + \overline{f}\, n & \geq & -\overline{f}\, n^0 \\ -f + \overline{f}\, n & \geq & -\overline{f}\, n^0 \end{array}$$

In the other hand, inequalities (4.23) and (4.24) can be partitioned likewise as

$$\begin{aligned} -g &\geq -\overline{g} \\ g &\geq 0 \\ -n &\geq -\overline{n} \\ n &\geq 0 \end{aligned}$$

Finally, the model is

$$minimize \quad v = c^T n$$

subject to :

$$\begin{aligned} \mathcal{S}f + g &= d \\ f + \overline{f}\,n &\geq -\overline{f}\,n^0 \\ -f + \overline{f}\,n &\geq -\overline{f}\,n^0 \\ -g &\geq -\overline{g} \\ -n &\geq -\overline{n} \\ g &\geq 0 \\ n &\geq 0 \\ n \in \mathbb{Z}^r&,\ f \in \mathbb{R}^n \end{aligned} \tag{4.25}$$

Note that this model is now a mixed-integer *linear* optimization problem (convex relaxation), and if we relax the integer constraints, the model becomes simply a linear optimization problem (continuous relaxation).

4.5.1 Garver's test system model development

The Garver's test system is a classic in studies of the transmission expansion planning [40]. Its usefulness is largely a question of a small system –whose mathematical model with few variables and constraints is very manageable– for which it is known the optimal solution; this makes it the most widely used test system for those who try to validate new algorithms and/or planning strategies.

In this subsection the problem statement is made for the Garver's system. Also, the transportation model given by (4.25) will be developed with and without generation redispatch.

Problem statement

Garver's system initially is a 5-bus system with 6 branches and 6 transmission circuits –one per branch[2]–. The system has a current demand $d = 190\,MW$ and a generating capacity $g = 270\,MW$ (Figure 4.3).

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[2] A right of way with at least one built circuit is called a *branch*.

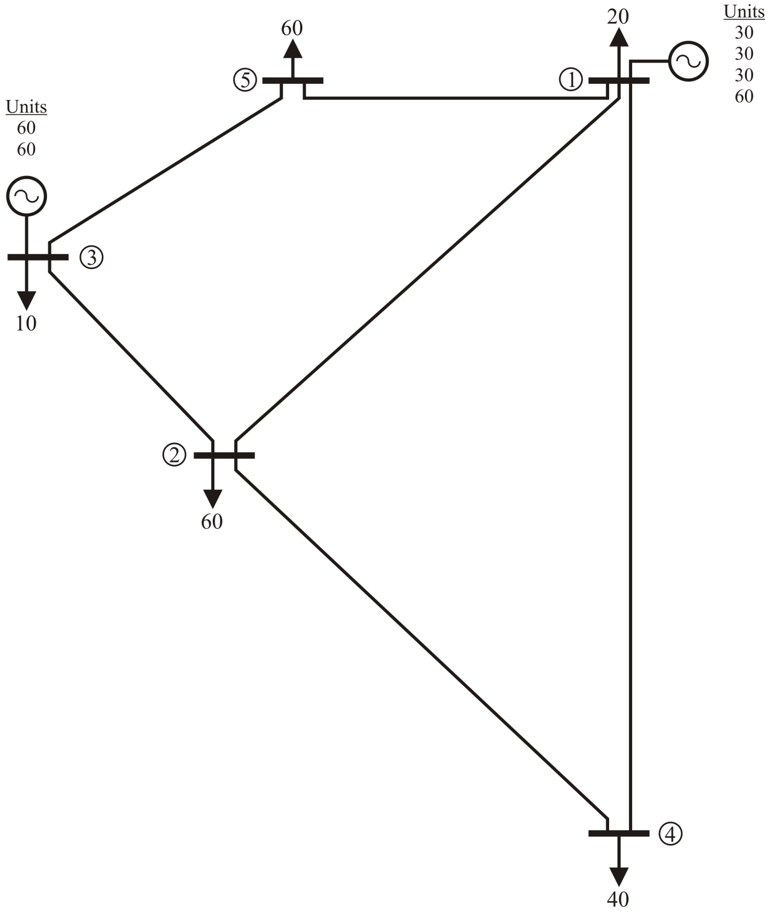

Figure 4.3: Original configuration of the Garver's test system.

Under these conditions, the system operates properly (Appendix C). In order to check all the system operation scenarios, we use an open source Matlab toolbox for electric power system analysis and simulation, the PSAT (Power System Analysis Toolbox) [58].

A future condition where demand will grow four times its present value and a new generation bus with a maximum generating capacity of 600 MW is expected. Also, it is planned a strengthening at bus three which consists of two more units of 120 MW each. The future system (which will be the *initial topology* for the STEP modeling) is as shown in Figure 4.4.

To integrate the new bus, the land linking bus 2 with bus 6 and bus 4 with bus 6 has been purchased, so the system stays with the following characteristics:

- Maximum demand: 760 MW
- Maximum generation capacity: 1110 MW
- ROW's: 1-2, 1-4, 1-5, 2-3, 2-4, 3-5 (current); 2-6 and 4-6 (new).

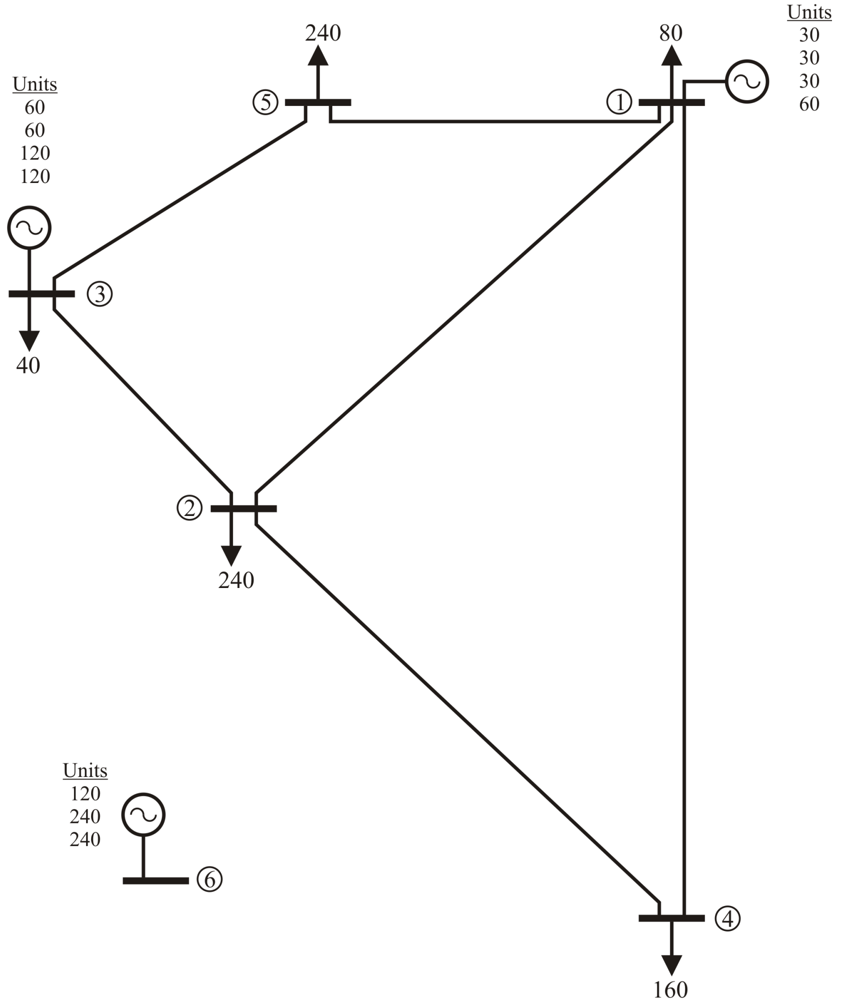

Figure 4.4: Future condition –initial topology for the STEP modeling– of Garver's system.

The maximum generation and load data of the future condition are shown in Table 4.1; ROW data are shown in Table 4.2.

Then, the question is: Where and how many transmission circuits must be built for a new configuration of the transmission network that minimizes the cost of construction and that satisfies the requirements of future conditions?

In order to answer the previous question, we can raise two schemes of modeling:

- One scheme which takes into account a previous generation dispatch where total generation is equal to demand (called study *without redispatch*); or
- another scheme where generation is greater than demand, allowing the model to yield a dispatch of the generation units (called study *with redispatch*).

Table 4.1: Generation and load data for Garver's system

Bus No.	$\overline{g}_i$ $[MW]$	d_i $[MW]$
1	150	80
2	—	240
3	360	40
4	—	160
5	—	240
6	600	—
Total	1110	760

Table 4.2: Right of way data for Garver's system

Circuit	n_{ik}^0	**Cost,** $\times 10^3$ $[USD]$	$\overline{f}_{ik}$ $[MW]$
1-2	1	40	100
1-4	1	60	80
1-5	1	20	100
2-3	1	20	100
2-4	1	40	100
3-5	1	20	100
2-6	0	30	100
4-6	0	30	100

STEP Transportation Model Without Redispatch (WOR)

In a STEP without redispatch, a previous generation dispatch for the future demand conditions is defined. In this work, we take the dispatch generation proposed in [40] totaling the 760 MW of demand given by:

$$g_1 = 50\,MW,\ g_3 = 165\,MW \text{ and } g_6 = 545\,MW$$

to have the following model development.

Objective function. Taking the cost of construction of new circuits data of Table 4.2, the objective function is:

$$min \quad v = 40n_{12} + 60n_{14} + 20n_{15} + 20n_{23} + 40n_{24} + 20n_{35} + 30n_{26} + 30n_{46}$$

Equality constraints. The *node-branch* incidence matrix is

$$\mathcal{S} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Then, the power balance equation (4.21) is

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} f_{12} \\ f_{14} \\ f_{15} \\ f_{23} \\ f_{24} \\ f_{35} \\ f_{26} \\ f_{46} \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \\ 165 \\ 0 \\ 0 \\ 545 \end{bmatrix} = \begin{bmatrix} 80 \\ 240 \\ 40 \\ 160 \\ 240 \\ 0 \end{bmatrix}$$

According to the expression above, some notes concerning the modeling of the balance equations are foreseen:

For bus 1: The generation of 50 MW will be taken to meet the demand of 80 MW at the bus and this bus will be modeled as a 30 MW load bus.

For bus 3: Part of the 165 MW generated will supply the 40 MW of demand and this bus will be modeled as a generation bus with maximum limit of 125 MW.

Considering the latter and taking *p.u.* values, the previous matrix expression can be written as follows

$$\begin{aligned} -f_{12} - f_{14} - f_{15} &= 0.30 \\ f_{12} - f_{23} - f_{24} - f_{26} &= 2.40 \\ f_{23} - f_{35} + g_3 &= 0 \\ f_{14} + f_{24} - f_{46} &= 1.60 \\ f_{15} + f_{35} &= 2.40 \\ f_{26} + f_{46} + g_6 &= 0 \end{aligned}$$

Inequality constraints. The maximum power flow limits given by (4.22) is expanded as

$$\begin{aligned} f_{12} + n_{12} &\geq -1 \\ -f_{12} + n_{12} &\geq -1 \\ f_{14} + n_{14} &\geq -0.8 \\ -f_{14} + n_{14} &\geq -0.8 \\ f_{15} + n_{15} &\geq -1 \\ -f_{15} + n_{15} &\geq -1 \\ f_{23} + n_{23} &\geq -1 \\ -f_{23} + n_{23} &\geq -1 \\ f_{24} + n_{24} &\geq -1 \\ -f_{24} + n_{24} &\geq -1 \\ f_{35} + n_{35} &\geq -1 \\ -f_{35} + n_{35} &\geq -1 \end{aligned}$$

$$\begin{aligned} f_{26}+n_{26} &\geq 0 \\ -f_{26}+n_{26} &\geq 0 \\ f_{46}+n_{46} &\geq 0 \\ -f_{46}+n_{46} &\geq 0 \end{aligned}$$

The maximum power generation limits (4.23) are written as

$$\begin{aligned} -g_3 &\geq -1.25 \\ -g_6 &\geq -5.45 \\ g_3 &\geq 0 \\ g_6 &\geq 0 \end{aligned}$$

The positivity on the transmission circuits addition are

$$\begin{aligned} n_{12} &\geq 0 \\ n_{14} &\geq 0 \\ n_{15} &\geq 0 \\ n_{23} &\geq 0 \\ n_{24} &\geq 0 \\ n_{35} &\geq 0 \\ n_{26} &\geq 0 \\ n_{46} &\geq 0 \end{aligned}$$

Note that we have not considered upper bounds in the space for construction, *i.e.*, $n \in \mathbb{Z}_+^r$.

Thus, the transportation model for the Garver's system –including slack variables– is:

$$\textit{minimize} \quad v = 40n_{12}+60n_{14}+20n_{15}+20n_{23}+40n_{24}+20n_{35}+30n_{26}+30n_{46}$$

subject to :

$$\begin{aligned} -f_{12}-f_{14}-f_{15} &= 0.30 \\ f_{12}-f_{23}-f_{24}-f_{26} &= 2.40 \\ f_{23}-f_{35}+g_3 &= 0 \\ f_{14}+f_{24}-f_{46} &= 1.60 \\ f_{15}+f_{35} &= 2.40 \\ f_{26}+f_{46}+g_6 &= 0 \\ f_{12}+n_{12}-s_1 &= -1 \\ -f_{12}+n_{12}-s_2 &= -1 \\ f_{14}+n_{14}-s_3 &= -0.8 \\ -f_{14}+n_{14}-s_4 &= -0.8 \\ f_{15}+n_{15}-s_5 &= -1 \\ -f_{15}+n_{15}-s_6 &= -1 \end{aligned} \tag{4.26}$$

$$\begin{aligned}
f_{23} + n_{23} - s_7 &= -1 \\
-f_{23} + n_{23} - s_8 &= -1 \\
f_{24} + n_{24} - s_9 &= -1 \\
-f_{24} + n_{24} - s_{10} &= -1 \\
f_{35} + n_{35} - s_{11} &= -1 \\
-f_{35} + n_{35} - s_{12} &= -1 \\
f_{26} + n_{26} - s_{13} &= 0 \\
-f_{26} + n_{26} - s_{14} &= 0 \\
f_{46} + n_{46} - s_{15} &= 0 \\
-f_{46} + n_{46} - s_{16} &= 0 \\
-g_3 - s_{17} &= -1.25 \\
-g_6 - s_{18} &= -5.45 \\
g_3 - s_{19} &= 0 \\
g_6 - s_{20} &= 0 \\
n_{12} - s_{21} &= 0 \\
n_{14} - s_{22} &= 0 \\
n_{15} - s_{23} &= 0 \\
n_{23} - s_{24} &= 0 \\
n_{24} - s_{25} &= 0 \\
n_{35} - s_{26} &= 0 \\
n_{26} - s_{27} &= 0 \\
n_{46} - s_{28} &= 0
\end{aligned}$$

$$n \in \mathbb{Z}_+^r,\ s \in \mathbb{R}_+^n \text{ and } f \in \mathbb{R}^n$$

We have the following characteristics to the STEP model for the Garver's test system without redispatch:

- Number of constraints: 34 (6 equality constraints and 28 inequality constraints)
  - 6 power balance constraints (one for each bus)
  - 16 maximum flow constraints (two per ROW)
  - 4 maximum generation constraints (two for each generation bus)
  - 8 positivity transmission circuit addition constraints (one for each ROW)
- Number of variables: 46
  - 8 variables associated to the circuit addition (n)
  - 8 variables related to the power flow in circuits (f)
  - 2 variables corresponding to the power generation (g)
  - 28 slack variables (s); one per inequality constraint

STEP Transportation Model With Redispatch (WR)

In this case generation is greater than demand, *i.e.*, no previous dispatch is considered, instead the maximum generation capacity of each plant will be used in a free manner in order to satisfy the load. We will show the necessary modifications when the model is considering redispatch.

First, the power balance equation (4.21) is

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} f_{12} \\ f_{14} \\ f_{15} \\ f_{23} \\ f_{24} \\ f_{35} \\ f_{26} \\ f_{46} \end{bmatrix} + \begin{bmatrix} g_1 \\ 0 \\ g_3 \\ 0 \\ 0 \\ g_6 \end{bmatrix} = \begin{bmatrix} 80 \\ 240 \\ 40 \\ 160 \\ 240 \\ 0 \end{bmatrix}$$

In this time, we have to consider the following remarks for the modeling of the balance equations:

For bus 1: This bus with 150 MW of maximum generation available will fulfill the 80 MW of demand; thus, this bus will be modeled as a generation bus with maximum generation capacity of 70 MW.

For bus 3: This bus with 360 MW of maximum generation available will satisfy the 40 MW of demand; thus, this bus will be modeled as a generation bus with maximum generation capacity of 320 MW.

For bus 6: This bus will be modeled as a generation bus with maximum generation capacity of 600 MW.

Considering the latter and taking *p.u.* values, the previous matrix expression can be written as follows

$$\begin{aligned} -f_{12} - f_{14} - f_{15} + g_1 &= 0 \\ f_{12} - f_{23} - f_{24} - f_{26} &= 2.40 \\ f_{23} - f_{35} + g_3 &= 0 \\ f_{14} + f_{24} - f_{46} &= 1.60 \\ f_{15} + f_{35} &= 2.40 \\ f_{26} + f_{46} + g_6 &= 0 \end{aligned}$$

The maximum power generation limits (4.23) are written as

$$\begin{aligned} -g_1 &\geq -0.70 \\ -g_3 &\geq -3.2 \\ -g_6 &\geq -6.0 \\ g_1 &\geq 0 \\ g_3 &\geq 0 \\ g_6 &\geq 0 \end{aligned}$$

Note that the model finishes with 30 inequality constraints (totaling 36 when including the 6 equality constraints), 1 more variable (g_1) and 2 more slacks (totaling 49 variables).

4.5.2 24-bus IEEE test system

The system consists of 24 buses and 32 generation units injecting power in 10 buses with a current demand $d = 2\,850\,MW$ and a generating capacity $g = 3\,405\,MW$. However, it is expected an expansion to a future condition with the generation levels and the loads three times their original values, *i.e.*, $8\,550\,MW$ peak demand and up to a total of $10\,215\,MW$ maximum generation capacity (Figure 4.5).

Besides the 34 existing branches (with 38 circuits constructed), 7 new right of ways has been purchased, totaling 41 right of ways. All data for this test system is given in the Appendix D.

Nevertheless there appears to be enough generation capacity, a load shedding study shows a load curtailment suggesting a lack in circuits constructed [94].

When rescheduling is allowed, the model has the following characteristics:

- Number of constraints: 165
- Number of variables: 232

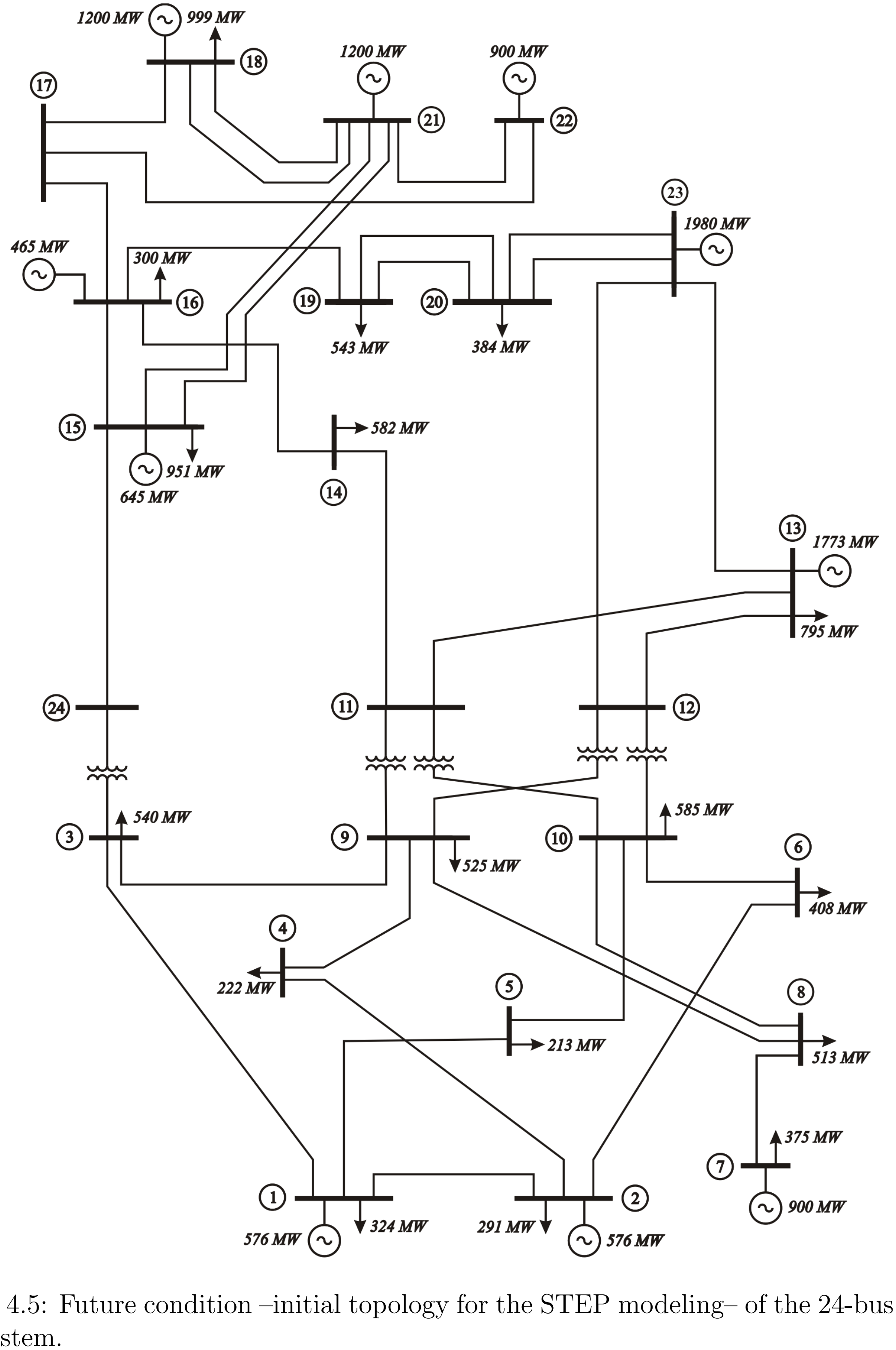

Figure 4.5: Future condition –initial topology for the STEP modeling– of the 24-bus IEEE test system.

Chapter 5

Solution Algorithm And Results

The results for the 6-bus Garver's test system and for the 24-bus IEEE test system are reported in this chapter. We give the solution with and without redispatch for the Garver's system and for the 24-bus system only the solution with redispatch is given. First we explain the algorithm used to handle the integer part of the problem. Then, we show the results of the sensitivity index used in the algorithm for all the expansion process; the optimal vector solution and the barrier parameter for the last iteration of the algorithm are shown as well. Finally, the optimal configuration of the network topology is given in each case of study.

As it was aforementioned in Chapter 1, we can classify the modeling approaches of the STEP problem as convex relaxed and continuous relaxed, and when a continuous relaxed modeling approach is performed, an heuristic to obtain the integer solutions are used in general [7, 17, 18, 19, 20, 21, 22].

The algorithm for the solution of the STEP problem used in this work is the Garver's constructive heuristic algorithm. A *Constructive Heuristic Algorithm* (CHA) is an iterative solution process designed to solve a specific complex problem with an acceptable quality by deciding the addition of one component of the solution at the time. The algorithm works using heuristic rules –represented by a *sensitivity index*–, searching for a good or acceptable solution in each iteration. The algorithm finishes when a reasonable and practical solution is found.

In the case of the STEP problem, the decision in each step of the process is circuit addition and is determined by the sensitivity index predefined; in fact, the major concern of a CHA is based on this index [21]. The literature shows that a CHA in STEP is robust and reach good quality solution with fast convergence, although this solution may be far from optimal [22]. However, when the problem is too large, the CHA could present problems with convergence [21].

5.1 Garver's Constructive Heuristic Algorithm

In the 70's, Garver was the first researcher who suggested the use of convex and continuous relaxation modeling of the STEP [40]. In his work, Garver presented the transportation model as a convex relaxation for the problem, and he dropped the integer constraints to obtain finally a continuous LP.

The Garver's CHA is based on the solution of the relaxed problem. Once the LP is solved, a set of values $\{n_{ik} \,|\, n_{ik} \geq 0 \;\; \forall\, (i,k) \in \mathcal{O}\}$[1] (non-integer in general) will be obtained indicating the need of addition (circuit construction) in at least one element of the set.

For example, if we relax the integer constraints, the solution of the LP formulation (4.26) for the circuit addition variables is:

$$\{n_{12} = 0,\, n_{14} = 0,\, n_{15} = 0.13,\, n_{23} = 0,\, n_{24} = 0,\, n_{35} = 0.27,\, n_{26} = 3.13,\, n_{46} = 2.32\}$$

What would be the best integer solution in this case? It is well known that in general, the rounding procedure to the nearest integer could fail [95]. Thus, which is the most attractive option for addition between all of this elements of the set? In other words, where should a circuit be added? Looking for a measure of the ROW's overload, the answer of Garver was: the circuit which is transporting the largest amount of power flow *i.e.*, the circuit with the biggest value of $n_{ik}\overline{f}_{ik}$.

Therefore, the Garver's CHA sensitivity index is given by:

$$SI = max\{n_{ik}\overline{f}_{ik}\} \tag{5.1}$$

The addition will produce a reconfiguration of the topology in every step, and this should be repeated till no circuit addition is necessary, *i.e.*, until $n_{ik} = 0$, or equivalently until the cost of construction given by the objective function of the problem (4.20) is zero ($v = 0$); we set a stopping parameter ϵ for this in the implementation (***cost of construction stopping criterion***).

Thus, considering the notation given in Chapter 4, the Garver's CHA for STEP is showed in Algorithm 4.

Note in the Algorithm 4 that in step three and nine, it is necessary to solve a LP. These LP will be solved by the IPM described in Chapter 3.

Observation 5.1 *It should be clear that since decision criterion is based on local performance –the largest amount of power flow in lines–, the optimal solution will lose the global context.*

---

[1] Here, $\mathcal{O}$ represents the set of all ROW's in the system. For instance, in the Garver test system $\mathcal{O} = \{(1,2),(1,4),(1,5),(2,3),(2,4),(3,5),(2,6),(4,6)\}$

Algorithm 4: Garver's Constructive Heuristic Algorithm

```
Data: $n^0$, $c$, $\mathcal{S}$, $g$, $d$, $\overline{f}$, $\overline{g}$ and $\epsilon$
Result: The optimal final topology of the transmission system $n^*$
begin
    Take the initial topology $n^0$;
    Consider a continuous relaxation and solve (4.25) as a continuous LP using the
      Algorithm 3 gave in section 3.7;
    Compute the cost of construction $v$ (the value of the objective function of
      (4.25));
    while $v > \epsilon$ do
        Identify the most attractive ROW $(i - k)$ using the sensitivity index (5.1);
        Add a circuit in the corresponding ROW: $n^0 = [\ldots, n^0_{ik} + 1, \ldots]$;
        Update the network topology;
        Solve (4.25) as a continuous LP using the algorithm 3;
        Compute $v$;
    end
    $n^* \leftarrow n^0$;
end
```

5.2 Results

We show the optimal solution for the transmission expansion of the two test systems studied in this work. In both cases, the optimal answer to the questions *where to build*? and *how many circuits*? is given and was obtained by the CHA with the infeasible IPM inside to it; all the cases were coded and simulated in a MATLAB R2013a environment, running in an HP Compaq 8710w Mobile Workstation[2] with the following considerations in every situation:

- Starting solution $w^0 = [1, 1, \ldots, 1]$
- Non-symmetric form of the search direction expression given by the equation (3.17)
- For the step length (3.20), we use $\alpha_0 = 0.95$
- The reduction of the barrier parameter was calculated by equation (3.26) with a centering parameter of $\sigma = 0.1$
- For the cost of construction stopping criterion we use $\epsilon = 10^{-4}$ for the Garver test system and $\epsilon = 10^{-3}$ for the 24-Bus IEEE test system

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[2] http://h18000.www1.hp.com/products/quickspecs/archives_Canada/12731_ca_v4/12731_ca.HTML

5.2.1 6-bus Garver's test system - WOR

In Table 5.1, and following the Algorithm 4, we show the computation of the SI for the 6-bus Garver's test system without redispatch. The processing time required to achieve the optimal expansion was 2.016 seconds.

Observation 5.2 *Some comments on the table 5.1:*

1. *The bolded numbers indicate the SI for every iteration and establish the circuit addition in the corresponding ROW.*
2. *The value of* $v = 171.5$ *(iteration* 0*) corresponds to the cost of construction of the STEP continuous LP for the initial topology, i.e., the solution of (4.26) (given in section 5.1).*
3. *After the sixth iteration we get* $v = 9.8882 \times 10^{-11}$*, showing that an addition is not longer required.*

Table 5.1: Sensitivity Index for the Garver's system WOR

	$n_{ik}^0\overline{f}_{ik}$						
Iteration	**0**	**1**	**2**	**3**	**4**	**5**	**6**
v **ROW**	171.5	141.5	111.5	81.5	51.5	21.5	13.5
1 − 2	0	0	0	0	0	0	0
1 − 4	0	0	0	0	0	0	0
1 − 5	13.2601	13.0996	12.5767	12.4502	12.3589	10.9815	0
2 − 3	0	0	0	0	0	0	0
2 − 4	0	0	0	0	0	0	0
3 − 5	26.7399	26.9004	27.4233	27.5498	27.6411	**29.0185**	0
2 − 6	**313.0008**	218.0750	**206.7492**	117.8274	**89.9419**	22.0357	**23.1591**
4 − 6	231.9992	**226.9250**	138.2508	**127.1726**	55.0581	22.9643	21.8409

Thus, the Garver's test system –when the STEP problem is formulated WOR– needs the construction of 7 new transmission lines according to the following list (see also Figure 5.1):

- 1 in a current ROW $3-5$;
- 4 in the new ROW $2-6$; and
- 2 in the new ROW $4-6$.

giving an **optimal construction cost** of 200×10^3 USD.

Therefore, according to the ROW data (table 4.2), $n^0 = [1\ 1\ 1\ 1\ 1\ 1\ 0\ 0]$. After the expansion process, $n = [1\ 1\ 1\ 1\ 1\ 2\ 4\ 2]$.

The optimal solution vector –with 80 components– for the last iteration of the CHA is given in table 5.2, where we can see:

- the null values for the eight variables representing circuits addition (x_1 to $x_8 \rightarrow n_{12}$ to n_{46}), indicating that a growth is not longer required;
- the eight values corresponding to the power flows in ROW's (x_9 to $x_{16} \rightarrow f_{12}$ to f_{46});
- the two variables which stand for the generation buses ($x_{17} \rightarrow g_3$ and $x_{18} \rightarrow g_6$);
- the twenty eight unknowns corresponding to the slack variables for each inequality constraint (in this case we have used the command *format LONG* in order to have more accuracy);
- the six Lagrange multipliers for the equality constraints (λ_1 to λ_6); and
- the twenty eight Lagrange multipliers for the inequality constraints (π_1 to π_{28}).

Table 5.2: Optimal vector solution for the last iteration of the CHA for the Garver's tests system WOR

i	x_i	s_i	λ_i	π_i
1	0	0.4056	4.3580	0
2	0	1.5944	4.3580	0
3	0	0.4158	4.3580	0
4	0	1.1842	4.3580	0
5	0	1.6786	4.3580	0
6	0	0.3214	4.3580	0
7	0	1.4714	–	0
8	0	0.5286	–	0
9	-0.5944	1.2343	–	0
10	-0.3842	0.7657	–	0
11	0.6786	3.7214	–	0
12	0.4714	0.2786	–	0
13	0.2343	0.2999	–	0
14	1.7214	7.7001	–	0
15	-3.7001	0.2501	–	0
16	-1.7499	3.7499	–	0
17	1.25	2.169×10^{-12}	–	4.3580
18	5.45	2.066×10^{-12}	–	4.3580
19	–	1.25000	–	0
20	–	5.4500	–	0
21	–	3.09×10^{-13}	–	40
22	–	2.06×10^{-13}	–	60
23	–	6.17×10^{-13}	–	20
24	–	6.18×10^{-13}	–	20
25	–	3.09×10^{-13}	–	40
26	–	6.17×10^{-13}	–	20
27	–	4.13×10^{-13}	–	30
28	–	4.12×10^{-13}	–	30

Finally, in Figure 5.2 we show the behavior of the barrier parameter μ vs. iterations, for the final LP which solves the last iteration of the expansion process.

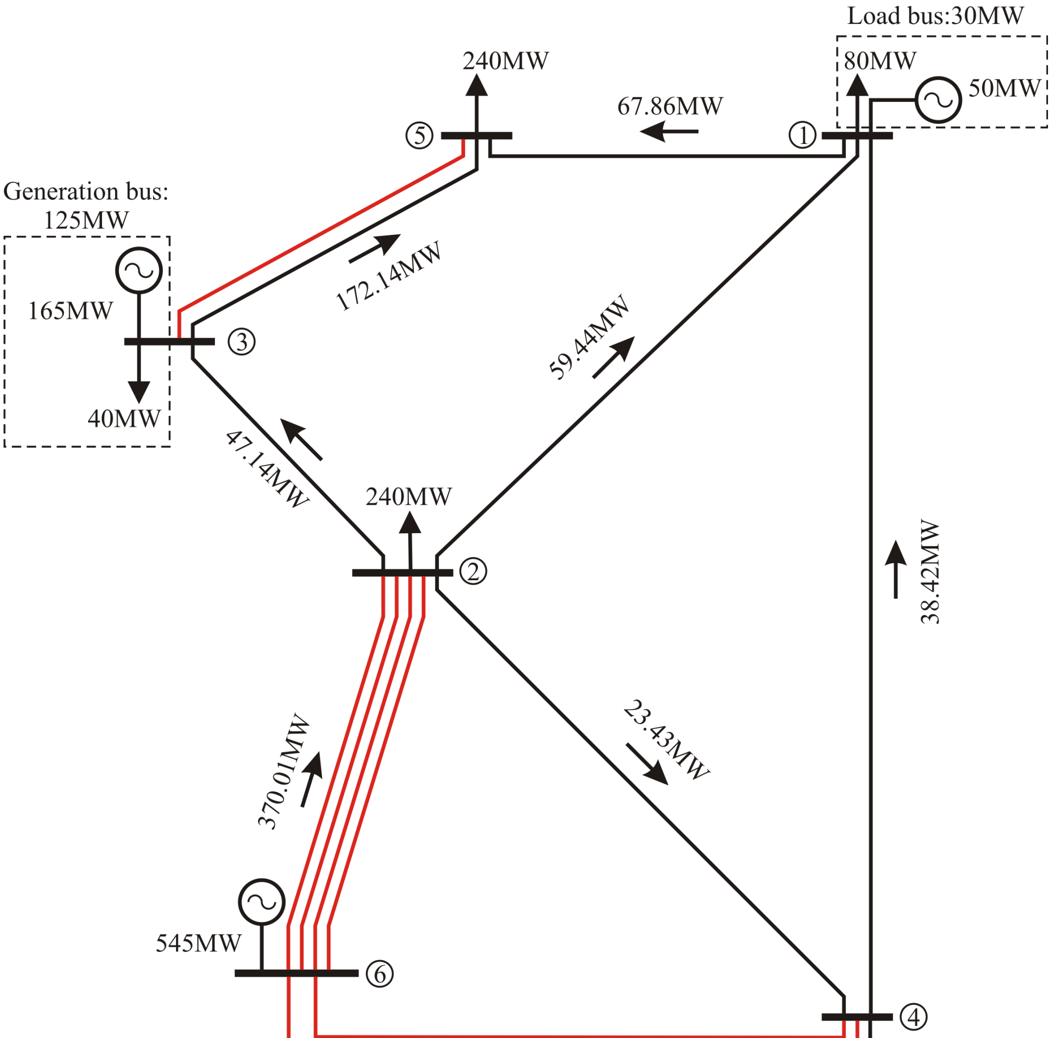

Figure 5.1: Final topology of Garver's test system WOR.

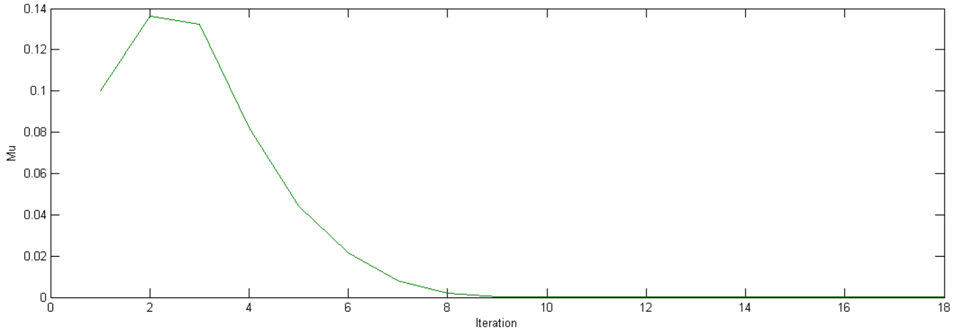

Figure 5.2: Behavior of the barrier parameter for the final iteration of the Garver's system WOR.

5.2.2 6-bus Garver's test system - WR

Similarly, in table 5.3 we show the computation of the SI for the 6-bus Garver's test system but with redispatch. The processing time required to achieve the optimal expansion was 1.856 seconds.

Table 5.3: Sensitivity Index for the Garver's system WR

	$n^0_{ik}\overline{f}_{ik}$				
Iteration	**0**	**1**	**2**	**3**	**4**
v / **ROW**	99.0	69.0	39.0	19.0	5.2738×10^{-12}
1 – 2	0	0	0	0	–
1 – 4	0	0	0	0	–
1 – 5	0	0	0	0	–
2 – 3	32.8736	35.7575	36.1930	9.5005	–
2 – 4	0	0	0	0	–
3 – 5	87.1264	84.2425	**83.8070**	10.4995	–
2 – 6	113.2743	**93.6740**	23.3715	23.9503	–
4 – 6	**136.7257**	56.3260	26.6285	**26.0497**	–

The Garver's test system –when the STEP problem is formulated WR– needs the construction of 4 new transmission lines according to the following list (see also Figure 5.3):

- 1 in a current ROW $3 - 5$;
- 1 in the new ROW $2 - 6$; and
- 2 in the new ROW $4 - 6$.

giving an **optimal construction cost** of 110×10^3 USD.

Again, according to the ROW data for this problem, $n^0 = [1\ 1\ 1\ 1\ 1\ 1\ 0\ 0]$. After the expansion process, $n = [1\ 1\ 1\ 1\ 1\ 2\ 1\ 2]$.

Under this configuration considering redispatch, the optimal solution vector –with 85 elements– for the last iteration of the CHA is given in table 5.4

Table 5.4: Optimal vector solution for the last iteration of the CHA for the Garver's tests system WR

i	x_i	s_i	λ_i	π_i
1	0	1.2444	0	0
2	0	0.7556	0	0
3	0	0.7549	0	0
4	0	0.8451	0	0
5	0	1.4659	0	0
6	0	0.5341	0	0
7	0	0.0668	–	0
8	0	1.9332	–	0
9	0.2444	0.7122	–	0
10	-0.0451	1.2878	–	0
11	0.4659	3.9341	–	0
12	-0.9332	0.0659	–	0
13	-0.2878	0.0654	–	0
14	1.9341	1.9346	–	0
15	-0.9346	0.0671	–	0
16	-1.9329	3.9329	–	0
17	0.6652	0.0348	–	0
18	2.8673	0.3327	–	0
19	2.8675	3.1325	–	0
20	–	0.6652	–	0
21	–	2.8673	–	0
22	–	2.8675	–	0
23	–	1.6×10^{-14}	–	40
24	–	1.1×10^{-14}	–	60
25	–	3.3×10^{-14}	–	20
26	–	3.2×10^{-14}	–	20
27	–	1.6×10^{-14}	–	40
28	–	3.4×10^{-14}	–	20
29	–	2.2×10^{-14}	–	30
30	–	2.2×10^{-14}	–	30

Finally, in Figure 5.4 we show the behavior of the barrier parameter μ vs. iterations, for the final LP which solves the last iteration of the expansion process.

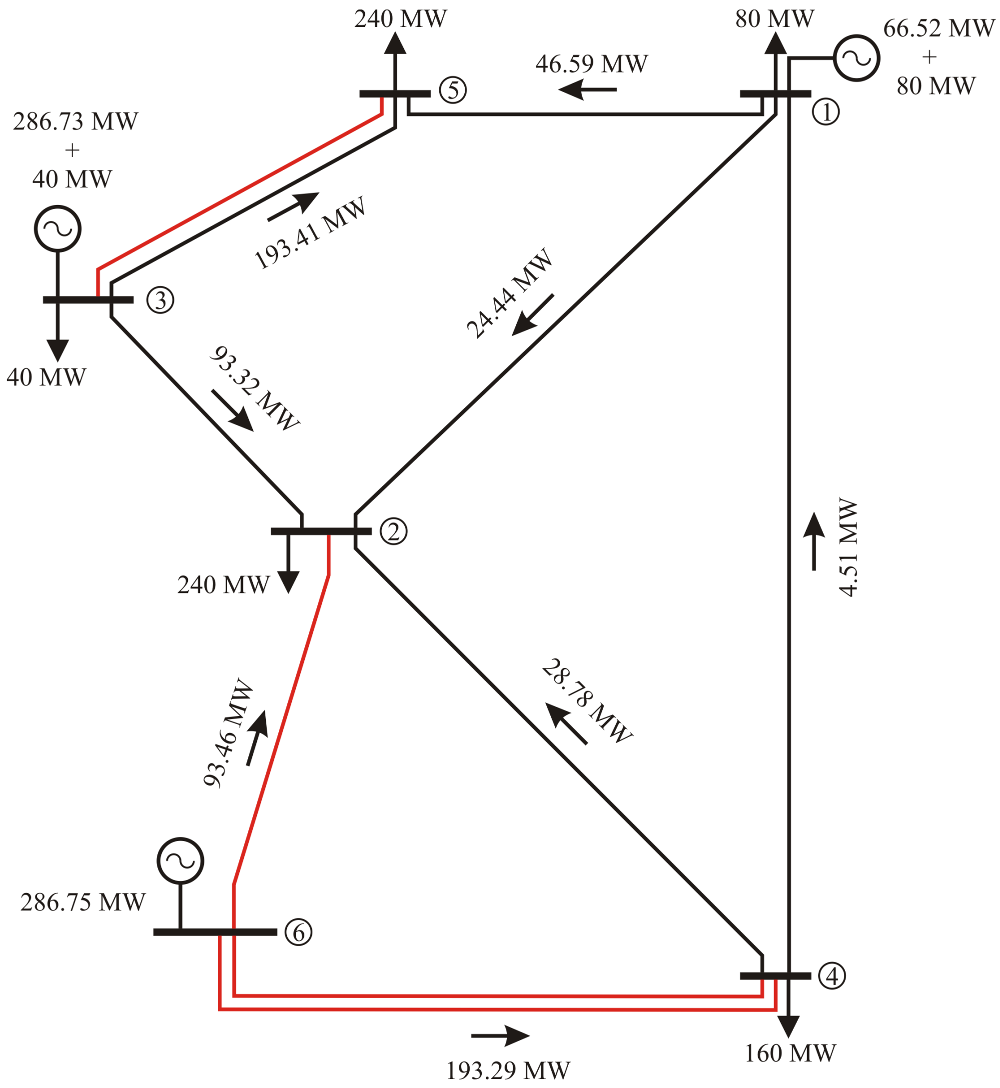

Figure 5.3: Final topology of Garver's test system WR.

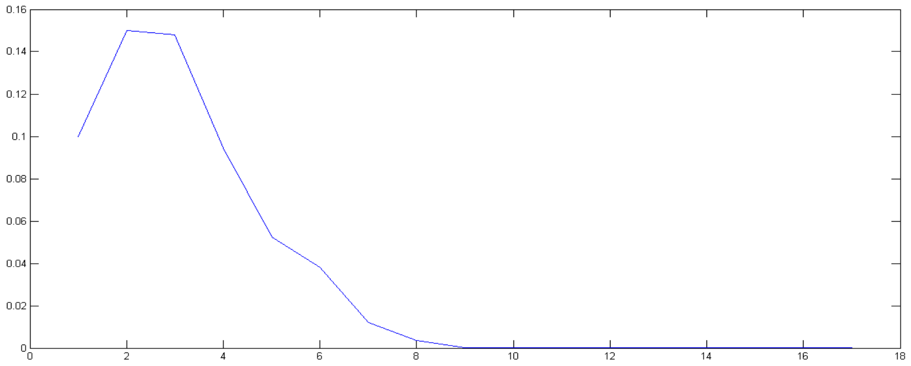

Figure 5.4: Behavior of the barrier parameter for the final iteration of the Garver's system WR.

5.2.3 24-bus IEEE test system - WR

In Table 5.5, and following the Algorithm 4, we show the computation of the SI for the 24-bus IEEE test system with redispatch. The processing time required to achieve the optimal expansion was 23.499 seconds.

The 24-bus IEEE test system –when the STEP problem is formulated WR– needs the construction of 4 new transmission lines according to the following list (see also Figure 5.5):

- 1 in the current ROW $6-10$;
- 2 in the current ROW $7-8$; and
- 1 in a new ROW $14-16$.

giving an **optimal construction cost** of 102×10^6 USD.

Again, according to the ROW data for this problem

$$n^0 = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$$

After the expansion process,

$$n = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 3\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$$

.

Finally, under this configuration considering redispatch, the optimal solution vector –with 397 elements– for the last iteration of the CHA is given in Table 5.6.

Table 5.5: Sensitivity Index for the 24-bus IEEE test system WR

	$n^0_{ik}\overline{f}_{ik}$				
Iteration	**0**	**1**	**2**	**3**	**4**
v **ROW**	72.5112	56.5111	16.1831	5.3031	2.0855×10^{-4}
1 − 2	0.0003	0.0003	0.0002	0.0002	–
1 − 3	0	0	0	0	–
1 − 5	0	0	0	0	–
2 − 4	0	0	0	0	–
2 − 6	0	0	0	0	–
3 − 9	0	0	0	0	–
3 − 24	0.0003	0.0003	0.0001	0	–
4 − 9	0	0	0	0	–
5 − 10	0	0	0	0	–
6 − 10	58.0	58.0	58.0	**58.0**	–
7 − 8	**349.9997**	174.9997	**118.9998**	0	–
8 − 9	0	0	0	0	–
8 − 10	0	0	0	0	–
9 − 11	0	0	0.0001	0	–
9 − 12	0.0003	0.0001	0	0	–
10 − 11	0	0	0.0001	0	–
10 − 12	0.0003	0	0.0001	0	–
11 − 13	0.0002	0	0	0	–
11 − 14	0	0	0	0	–
12 − 13	0	0	0	0	–
12 − 23	0	0	0	0	–
13 − 23	0	0	0	0	–
14 − 16	325.9991	**325.9991**	0	0	–
15 − 16	0.0001	0.0001	0.0001	0.0001	–
15 − 21	0	0	0	0	–
15 − 24	0	0	0	0	–
16 − 17	0.0001	0.0001	0.0001	0.0001	–
16 − 19	0.0001	0.0001	0.0001	0.0001	–
17 − 18	0.0001	0.0001	0.0001	0.0001	–
17 − 22	0	0	0	0	–
18 − 21	0.0001	0.0001	0.0001	0.0001	–
19 − 20	0	0	0	0	–
20 − 23	0.0001	0.0001	0.0001	0.0001	–
21 − 22	0	0	0	0	–
1 − 8	0.0001	0.0001	0.0001	0.0001	–
2 − 8	0.0001	0.0001	0.0001	0.0001	–
6 − 7	0.0001	0.0001	0.0001	0	–
13 − 14	0.0006	0.0006	0.0001	0.0001	–
14 − 23	0.0002	0.0002	0.0001	0.0001	–
16 − 23	0.0001	0.0001	0.0001	0	–
19 − 23	0.0001	0.0001	0.0001	0.0001	–

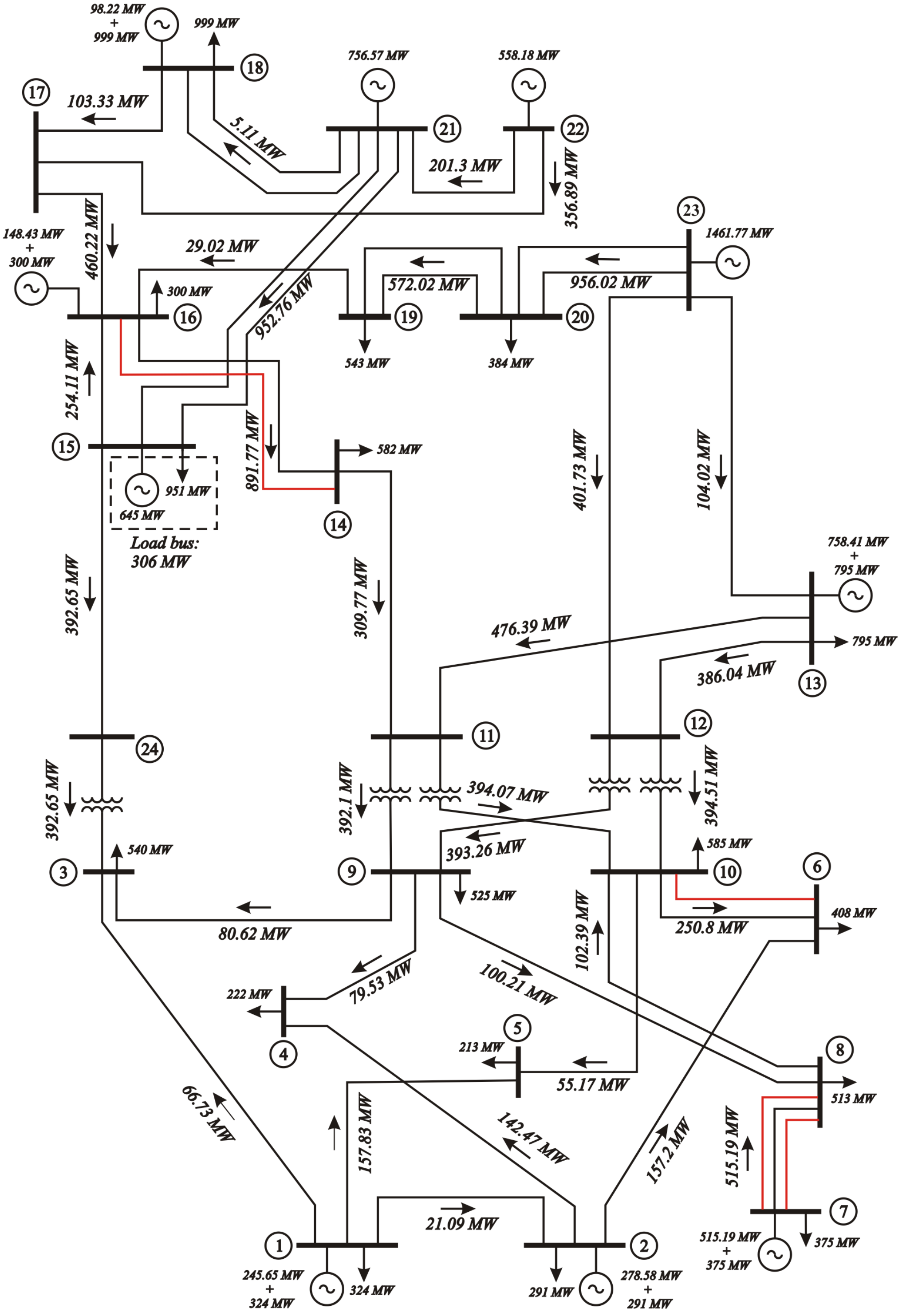

Figure 5.5: Final topology for the 24-bus IEEE test system WR.

Table 5.6: Optimal vector solution for the last iteration of the CHA for the 24-bus IEEE test system WR

i	x_i	s_i	λ_i	π_i	i	x_i	s_i	λ_i	π_i
1	0	1.9609	0	0	72	-0.0511	5.9563×10^{-7}	–	8.1825
2	0	1.5391	0	0	73	-5.7202	3.8333×10^{-7}	–	12.7926
3	0	2.4173	0	0	74	-9.5602	3.8295×10^{-7}	–	12.7925
4	0	1.0827	0	0	75	-2.0130	8.1819×10^{-7}	–	5.8188
5	0	3.3283	0	0	76	0	8.2007×10^{-7}	–	5.8188
6	0	0.1717	0	0	77	0	5.7028×10^{-7}	–	8.1775
7	0	3.1747	0	0	78	0	5.6949×10^{-7}	–	8.1775
8	0	0.3253	0	0	79	0	4.1719×10^{-7}	–	10.9346
9	0	3.3220	0	0	80	0	4.1705×10^{-7}	–	10.9346
10	0	0.1780	0	0	81	0	5.8587×10^{-7}	–	7.9695
11	0	0.9438	0	0	82	0	5.8540×10^{-7}	–	7.9695
12	0	2.5562	0	0	83	2.4565	0.0635	–	0
13	0	0.0735	0	0.0001	84	2.7858	0.0642	–	0
14	0	7.9265	0	0	85	5.1519	0.0981	–	0
15	0	0.9547	0	0	86	7.5841	2.1959	–	0
16	0	2.5453	0	0	87	1.4843	0.1657	–	0
17	0	1.1983	0	0.0001	88	0.9822	1.0278	–	0
18	0	2.3017	0	0	89	7.5657	4.4343	–	0
19	0	0.9920	0	0	90	5.5818	3.4182	–	0
20	0	6.0080	0	0	91	14.6177	5.1823	–	0
21	0	10.4019	0	0	92	–	2.4565	–	0
22	0	0.0981	0	0	93	–	2.7858	–	0
23	0	0.7479	0	0	94	–	5.1519	–	0
24	0	2.7521	0	0	95	–	7.5841	–	0
25	0	2.7739	–	0	96	–	1.4843	–	0
26	0	0.7261	–	0	97	–	0.9822	–	0
27	0	0.0790	–	0	98	–	7.5657	–	0
28	0	7.9210	–	0	99	–	5.5818	–	0
29	0	0.0674	–	0	100	–	14.6177	–	0
30	0	7.9326	–	0	101	–	1.3047×10^{-6}	–	3
31	0	0.0593	–	0	102	–	7.1022×10^{-8}	–	55
32	0	7.9407	–	0	103	–	1.7753×10^{-7}	–	22
33	0	0.0549	–	0	104	–	1.1841×10^{-7}	–	33
34	0	7.9451	–	0	105	–	7.8129×10^{-8}	–	50
35	0	0.2361	–	0	106	–	1.2602×10^{-7}	–	31
36	0	9.7639	–	0	107	–	7.8090×10^{-8}	–	50
37	0	1.9023	–	0	108	–	1.4468×10^{-7}	–	27
38	0	8.0977	–	0	109	–	1.6989×10^{-7}	–	23
39	0	1.1396	–	0	110	–	2.4415×10^{-7}	–	16
40	0	8.8604	–	0	111	–	2.4404×10^{-7}	–	16
41	0	0.9827	–	0	112	–	9.0842×10^{-8}	–	43
42	0.2109	9.0173	–	0	113	–	9.0845×10^{-8}	–	43
43	0.6673	3.9598	–	0.0001	114	–	7.8092×10^{-8}	–	50
44	1.5783	6.0402	–	0	115	–	7.8085×10^{-8}	–	50
45	1.4247	1.0823	–	0	116	–	7.8084×10^{-8}	–	50
46	1.5720	18.9177	–	0	117	–	7.8077×10^{-8}	–	50
47	-0.8062	7.5411	–	0.0001	118	–	5.9179×10^{-8}	–	66
48	-3.9265	2.4589	–	0	119	–	6.7358×10^{-8}	–	58
49	-0.7953	0.4724	–	0	120	–	5.9188×10^{-8}	–	66
50	-0.5517	19.5276	–	0	121	–	2.9150×10^{-8}	–	134
51	-2.5080	8.9265	–	0	122	–	3.2557×10^{-8}	–	120
52	5.1519	1.0735	–	0	123	–	7.2344×10^{-8}	–	54
53	-1.0021	0.3978	–	0	124	–	1.6285×10^{-7}	–	24
54	1.0239	9.6022	–	0	125	–	5.7446×10^{-8}	–	68
55	-3.9210	4.7098	–	0	126	–	5.4253×10^{-8}	–	72
56	-3.9326	5.2902	–	0	127	–	1.0850×10^{-7}	–	36
57	-3.9407	3.9667	–	0	128	–	1.2212×10^{-7}	–	32
58	-3.9451	6.0333	–	0	129	–	1.9537×10^{-7}	–	20
59	-4.7639	1.4311	–	0	130	–	2.6755×10^{-8}	–	146
60	-3.0977	8.5689	–	0	131	–	1.0852×10^{-7}	–	36
61	-3.8604	9.9489	–	0	132	–	7.1035×10^{-8}	–	55
62	-4.0173	10.0511	–	0	133	–	1.3020×10^{-7}	–	30
63	-1.0402	4.2798	–	0	134	–	4.1556×10^{-8}	–	94
64	-8.9177	15.7202	–	0	135	–	3.2390×10^{-7}	–	4.4560
65	2.5411	0.4398	–	0	136	–	3.4438×10^{-7}	–	4.3614
66	-9.5276	19.5602	–	0	137	–	2.2293×10^{-7}	–	5.2261
67	3.9265	2.9870	–	0	138	–	1.7128×10^{-7}	–	3.8119
68	-4.6022	7.0130	–	0	139	–	1.2143×10^{-7}	–	4.2253
69	-0.2902	5.5984×10^{-7}	–	8.7269	140	–	9.0881×10^{-8}	–	4.6543
70	-1.0333	5.5983×10^{-7}	–	8.7269	141	–	1.2458×10^{-7}	–	4.3052
71	-3.5689	5.9571×10^{-7}	–	8.1825	x	x	x	x	x

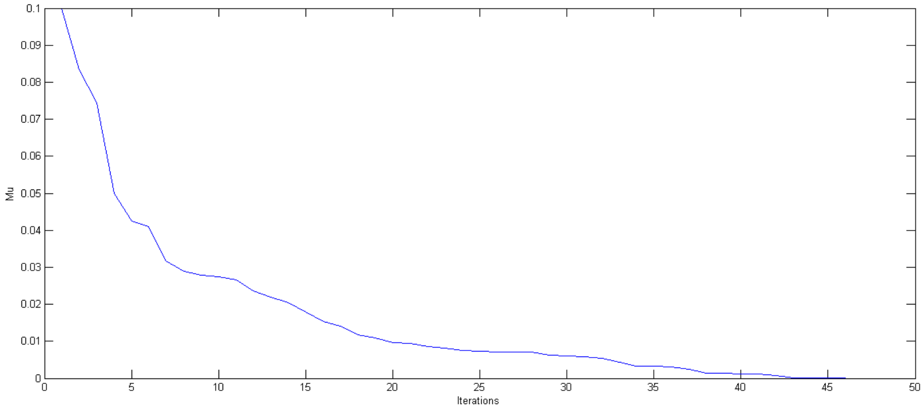

Figure 5.6: Behavior of the barrier parameter for the final iteration of the 24-bus IEEE test system.

Chapter 6

Conclusions and Future Research

The conclusions and some suggestions of future research are exposed in this part of the work.

6.1 Conclusions

We have studied the problem of STEP for two test systems, the 6-bus Garver's test system and the 24-bus IEEE test system. In both systems, satisfactory results which have already been reported in the literature were obtained.

The model described in this research deals with minimizing investment costs of transmission lines in the objective and with a transportation model for the network in the constraints. The modeling approaches presented in this work for solving the problem are: modeling with generation redispatch and without generation redispatch for the Garver's system, and modeling with generation redispatch for the 24-bus IEEE test system.

From the analysis of the results obtained for the two problems with and without redispatch of the Garver's system and considering the cost obtained for each study, it is clear that the redispatch model allows an expansion plan with a lower cost; this difference in costs (in many times a remarkable difference) has been continuously reported. The benefits of this scheme could not only be economic, because the generation capacity available to cover the demand allows better management of generation resources, resulting in efficient configurations of power flows on transmission lines. In addition to this, the results of the PSAT show that with redispatch scheme we obtain better profile of voltages in both magnitude and angle.

An important result in the estimation and distribution of flows in the system is observed, since in both schemes the estimations by the transportation model are very acceptable when compared with those obtained with the PSAT; even when the transportation model is a DC network modeling relaxation. This shows that the formulation is reliable for expansion stages at least in the long term, where the level of detail required is not very high.

It is important to add that the difference in the distribution of flows for the scheme without redispatch is due to the way in which the slack bus is chosen. For example, in the case of the Garver's system without redispatch, we choose the bus 6 as the slack bus (Table C.3), and if we take from the bus 6 generation ($P_{G_6} = 6.2214\ p.u.$) the system losses ($P_L = 0.7714\ p.u.$), we would obtain a generation of $P_{G_6} = 5.45\ p.u.$, which is reported for the transportation model and is shown in Figure 5.1.

Furthermore, since we deal with a mixed problem, a solution was proposed from the relaxation of the integer part in the model and the application of a CHA for the "rounding". Using this methodology, it is necessary to have a powerful solver able to solve in a quickly and reliable manner the linear (or nonlinear) program resulting from the relaxation. In this thesis we use an infeasible primal-dual interior point method, which is known in the literature as the most efficient of these methods. Talking about this, it is noteworthy that the behavior of the barrier parameter was almost the same for all iterations as it is shown in Figures 5.2, 5.4 and 5.6.

6.1.1 Contributions

The author strongly believed that the main contribution of this thesis is the explanation and exposure in detail of the transmission expansion planning problem, its related mathematics and its solution process using a CHA. As it can be seen along the work, all the topics were developed in a very comprehensive manner and in such a way that a future extension for more complicated models can be straightforward.

Added to this, another important contributions are:

- A clear classification of the STEP problem according to the mathematical modeling approaches (Chapter 1, section 1.1);
- Showing the connection between optimality conditions from the duality theory (2.6) and from the Karush-Kuhn-Tucker (2.8) points of view (Chapter 2, Observation 2.1);
- The full development of an Infeasible IPM and the detailed description of the algorithm (Chapter 3, Algorithm 3);
- In Chapter 4, obtaining the energy conservation equation (4.14) –which is part of the DC network model– from the AC transmission line power flow equations (4.3) and (4.4);
- Also in Chapter 4, obtaining the transportation model of the STEP problem (4.25) from the relaxation of the DC model (4.19) and a full description of its objective function and every constraint contained in the model;

- A detailed example of modeling of the STEP problem with and without redispatch (Chapter 4, subsection 4.5.1);
- The use of an IPM within the STEP problem as a solver in a Constructive Heuristic Algorithm and its detailed description (Chapter 5, Algorithm 4);
- Introducing a novel Power System Analysis Toolbox for the test systems operation condition verification (Appendix C).

The following two small contributions are results of this work as well:

1. International Poster
    - Becerril, C., Mota, R. and Badaoui, M. *Interior point algorithm as applied to the transmission network expansion planning problem.* SIAM Conference on optimization, San Diego, California, USA. May 19-22, 2014.
2. Conference paper
    - Becerril, C., Mota, R. and Badaoui, M. *Solution to the static transmission expansion planning by a primal-dual interior point method.* 7° Congreso Internacional de Ingeniería Electromecánica y de Sistemas, CIIES 2014.

6.2 Future Work

Since the model worked in this thesis meets minimizing the costs of construction and the requirement of satisfying the demand flows, possible aspects of operation and markets are not considered. That is why we suggest as a first step to add generation cost curves on the objective, and as second step to search strategies on which the market aspects such as congestion or market power can be taken into account.

On the other hand, the increasing loads and the large inter-utility power transfers are forcing systems to operate near their loadability limits, yielding many risk operating conditions. Thus, it would be very useful an improvement in the model in such a way that contingencies can be considered.

The last modification can lead to a more complete study where some problems as power flows unsolvability and voltage stability based on the expansion plan can be treated.

From the same philosophy of using the CHA for the STEP problem, it would be interesting to propose a non-linear formulation, where we can test the full development of the Vanderbei and Shanno's algorithm for nonlinear nonconvex programming problems.

In order to be able to face other temporal situation as the short term transmission expansion planning –where the necessity of a more detailed model is mandatory– we also suggest to work on the AC model.

Talking about the formulation and way of solving used in this work, it would be very interesting to propose and test some starting points techniques for the STEP problem.

Finally, an immediate work could be program the algorithm in other language in order to reduce the computation time obtained in this work.

Bibliography

[1] Gönen, T. *Electric power transmission system engineering: analysis and design*. CRC Press, Taylor and Francis Group, 3rd ed., (2014).

[2] Latorre, G., Cruz, R., Areiza, J. and Villegas, A. *Classification of publications and models on transmission expansion planning*. IEEE Trans. on Power Systems, Vol. 18, No. 2, pp. 938-946, May (2003).

[3] Lee, C., Simon, K., Zhong, J. and Wu, F. *Transmission expansion planning from past to future*. IEEE PES Power System Conference and Exposition, pp. 257-265, Oct. 29 - Nov. 1, (2006).

[4] Murty, K. *Operation research: Deterministic optimization models*. Prentice Hall, (1995).

[5] Oloomi, M., et al. *Transmission planning approaches in restructured power systems*. IEEE Bologna Power Tech Conference, June (2003).

[6] Oloomi, M. *Transmission expansion planning in deregulated power systems*. Ph. D. dissertation, Technischen Universität Darmstadt, Germany, (2004).

[7] Miranda, I., Silva, I. and Marcato, A. *Static planning of the expansion of electrical energy transmission systems using a particle swarm optimization*. Elsevier, Electrical Power and Energy Systems 60, pp. 234-244, (2014).

[8] Akbari, T. and Tavakoli, M. *A linearized formulation of AC multi-year transmission expansion planning: A mixed integer linear programming approach*. Elsevier, Electric Power System Research 114, pp. 93-100, (2014).

[9] Coffrin, C. and Van Hentenryck, P. *A linear-programming approximation of AC power flows*. INFORMS Journal of Computing, Vol. 26, No. 4, pp. 718-734, (2014).

[10] Zhang, H. et. al. *An improved network model for transmission expansion planning considering reactive power and network losses*. IEEE Trans. on Power Systems, Vol. 28, No. 3, pp. 3471-3479, August, (2013).

[11] Ahlhaus, P. and Stursberg, P. *Transmission capacity expansion: An improved transport model.* 4th IEEE PES Innovative Smart Grid Technologies Europe, Copenhagen, October 6-9, (2013).

[12] Zhang, H., Heydt, G. T., Vittal, V. and Mittelmann, H. D. *Transmission expansion planning using an AC model: formulations and possible relaxations.* Proc. IEEE PES General Meeting, July, (2012).

[13] Burer, S. and Letchford, A. *Non-convex mixed-integer nonlinear programming: A survey.* Surveys in Operation Research and Management Science 17, pp. 97-106, (2013).

[14] Huang, K-L. *Improving solvability of continuous and mixed-integer convex optimization problems using interior approaches.* Ph. D. dissertation, Northwestern University, August, (2012).

[15] Moulin, L., Poss, M. and Sagastizábal, C. *Transmission expansion planning with re-design.* Springer, Energy Systems, Vol. 1, No. 2, pp. 113-139, May (2010).

[16] Wanufelle, E. *A global optimization method for mixed integer nonlinear nonconvex problems related to power system analysis.* Ph. D. dissertation, Facultés Universitaires Notre-Dame de la Paix, (2007).

[17] Mahmoudabadi, A. et. al. *Implementation of an AC model for transmission expansion planning considering reliability constraints.* Turkish Journal of Electrical Engineering & Computer Sciences 21, pp. 1120-1133, (2013).

[18] Mahmoudabadi, A., et. al. *An application of CHA to concurrent short-term transmission expansion & reactive power planning.* IEEE Trondheim PowerTech, (2011).

[19] Rider, M. J., García, A. V. and Romero, R. *Power system transmission network expansion planning using AC model.* IET Gener. Transm. Distrib., Vol. 1, (5), pp. 731-742, (2007).

[20] Sánchez, I. G., Romero, R., Mantovani, J. R. S. and García, A. *Interior point algorithm for linear programming used in transmission network synthesis.* Elsevier, Electric Power System Research 76, pp. 9-16, July (2005).

[21] Sánchez, I. G., Romero, R., Mantovani, J. R. S. and Rider, M. J. *Transmission-expansion planning using the DC model and nonlinear-programming technique.* IEE Proc.-Gener. Transm. Distrib., Vol. 152, No. 6, pp. 763-769, November (2005).

[22] Romero, R., Rocha, C., Mantovani, M. and Mantovani, J. R. S. *Analysis of heuristic algorithms for the transportation model in static and multistage planning in network expansion systems.* IEE Proc.-Gener. Transm. Distrib., Vol. 150, No. 5, pp. 521-526, September, (2003).

[23] Alguacil, N., Motto, A. and Conejo, A. *Transmission expansion planning: A mixed-integer LP approach.* IEEE Trans. on Power Systems, Vol. 18, No. 3, August, (2003).

[24] Haffner, S., Monticelli, A., García, A. Mantovani, J. and Romero, R. *Branch and bound algorithm for transmission system expansion planning using a transportation model.* IEE Proc.-Gener. Transm. Distrib., Vol. 147, No. 3, pp. 149-156, May, (2000).

[25] Haffner, S., Monticelli, A., García, A. and Romero, R. *Specialised branch-and-bound algorithm for transmission network expansion planning.* IEE Proc.-Gener. Transm. Distrib., Vol. 148, No. 5, pp. 482-488, September, (2001).

[26] Vanderbei, R. and Shanno, D. *An interior-point algorithm for nonconvex nonlinear programming.* Computational Optimization and Applications 13, pp. 231-252, (1999).

[27] Hemmati, R., Hooshmand, R-A. and Khodabakshian, A. *State-of-the-art of transmission expansion planning: Comprehensive review.* Elsevier, Renewable and Sustainable Energy Reviews 23, pp. 312-319, (2013).

[28] Kishore, T. S. and Singal, S. K. *Optimal economic planning of power transmission lines: A review.* Elsevier, Renewable and Sustainable Energy Reviews 39, pp. 949-974, (2014).

[29] Quintana, V., Leite, G. and Medina-Palomo, J. *Interior-point methods and their applications to power systems: A classification of publications and software codes.* IEEE Trans. on Power Systems, Vol. 15, No. 1, pp. 170-176, February, (2000).

[30] Farhat, I. A. and El-Hawary, M. E. *Interior point methods application in optimum operational scheduling of electric power systems.* IET Gener. Transm. Distrib. Vol. 3, Iss. 11, pp. 1020-1029, (2009).

[31] Leite, G. *Nonlinear optimal power flow by interior and non-interior point methods.* Ph. D. dissertation, University of Waterloo, (1998).

[32] Mehrotra, S. *On the implementation of a primal-dual interior point method.* SIAM Journal on Optimization, vol. 2 (4), pp. 575-601, (1992).

[33] Carpenter, T. J., Lustig, I. J., Mulvey, J. M., and Shanno, D.F., *Higher-order predictor-corrector interior-point methods with applications to quadratic objectives.* SIAM Journal on Optimization, vol.3 (4), pp. 696-725, (1993).

[34] Gondzio, J. *Multiple centrality corrections in a primal-dual method for linear programming.* Computational Optimization and Applications 6, pp. 137-156, (1996).

[35] Rider, M. J., Castro, C. A., Bedriñana, M. F. and García, A. V. *Toward a fast and robust interior point method for power system applications*. IEE Proc.-Gener. Transm. Distrib. Vol. 151, No. 5, pp. 575-581, September, (2002).

[36] Capitanescu, F., Glavic, M., Ernst, D. and Wehenkel, L. *Interior-point based algorithms for the solution of optimal power flow problems*. Elsevier, Electric Power System Research 77, pp. 508-517, (2007).

[37] Frank, S., Steponavice, I. and Rebennack, S. *Optimal power flow: a bibliographic survey I*. Springer-Verlag, Energy Systems 3, pp. 221-258, (2012).

[38] Capitanescu, F. and Wehenkel, L. *Experiments with the interior-point method for solving optimal power flow problems*. Elsevier, Electric Power Systems Research 95, pp. 276-283, (2013).

[39] Chiang, N. *Structure-exploiting interior point methods for security constrained optimal power flow problems*. Ph. D. dissertation, University of Edinburgh, Scotland, April, (2013).

[40] Garver, L. *Transmission network estimation using linear programming*. IEEE Trans. on Power Apparatus and Systems, Vol. PAS-89, No. 7, September/October (1970).

[41] Correa, C., Bolaños R. and Garces, A. *Environmental transmission expansion planning using non-linear programming and evolutionary techniques*. IEEE (2012).

[42] Molina, F. *Priorización de la adición de elementos de transmisión en la expansión de un sistema eléctrico de potencia*. M. S. Thesis, electrical research and posgraduate studies section, ESIME-IPN, (1997).

[43] Díaz, S. *Herramienta para la planeación de la expansión de la transmisión en ambientes desregulados*. M. S. Thesis, electrical research and posgraduate studies section, ESIME-IPN, (2006).

[44] Anaya, L. *Planeación de la red de transmisión para centrales eoloeléctricas*. M. S. Thesis, electrical research and posgraduate studies section, ESIME-IPN, (2012).

[45] Zenón, E. *La regulación y el bienestar en la expansión de la red de transmisión eléctrica en un ambiente simulado*. Engineering Doctorate, Engeneering Faculty, UNAM, (2011).

[46] Madrigal, M., Aboytes, F. and Flores, R. *Transmission management, pricing and expansion planning in Mexico: Current status and perspectives*. IEEE (2006)

[47] Ávila, M. and Mota, R. *Issues affecting the electricity transmission and distribution system in Mexico*. IEEE (2008).

[48] Kantorovich, L. V. *Mathematical methods in the organization and planning of production.* Management Sciences 6, pp. 550-559, (1960).

[49] Dantzig, G. B. *Maximization of a linear function of variables subject to linear inequalities.* Activity Analysis of Production and Allocation. pp. 339-347, (1947).

[50] Klee, V. and Minty, G. J. *How good is the simplex algorithm?* Technical report, Washington University Department of Mathematics, (1972).

[51] Nemirovski, A. *Optimization I: Introduction to linear optimization.* Lecture Notes, Georgia Institute of Technology, (2013).

[52] Chong, E. and Żak, S. *An introduction to optimization.* Wiley, 4th ed., (2013).

[53] Khachiyan, L. G. *A polynomial algorithm for linear programming.* Soviet Math. Doklady, Vol. 20, pp. 191-194, (1979).

[54] Shano, D. *Who invented the interior-point method?* Documenta Mathematica, extra volume ISMP, pp. 55-64, (2012).

[55] Ye, Y. *Interior point algorithms: Theory and Analysis.* Wiley-Intersciences Series in Discrete Mathematics and Optimization, (1997).

[56] Karmakar, N. *A new polynomial-time algorithm for linear programming.* Combinatorica 4, pp. 373-395, (1984).

[57] Robere, R. *Interior point methods and linear programming.* University of Toronto, (2012).

[58] Milano, F. *An open source power system analysis toolbox.* IEEE Trans. on Power Systems, Vol. 20, No. 3, August, (2005).

[59] Andréasson, N., Evgrafov, A. and Patriksson, M. *An introduction to continuous optimization.* Studentlitteratur, Chalmers University of Technology, May, (2005).

[60] Kuhn, H. W. and Tucker, A. W. *Nonlinear programming.* Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. Neyman Editor, pp. 481-492. University of Californa Press, LA California, (1961).

[61] Karush, W. *Minima of functions of several variables with inequalities as side conditions.* Technical report, M. S. Thesis, Department of Mathematics, University of Chicago, (1939).

[62] Gondzio, J. *Interior point methods 25 years later.* Elsevier, European Journal of Operational Research Vol. 218, Issue 3, pp. 587-601, May, (2012).

[63] Caron, R. J., Greenberg, H. J. and Holder, A. G. *Analytic centers and repelling inequalities*. Elsevier, European Journal of Operational Research Vol. 143, pp. 268-290, (2002).

[64] Jansen, B., Roos, C. and Terlaky, T. *Introduction to the theory of interior point methods*. Interior Point Methods of Mathematical Programming, Applied Optimization Vol. 5, pp. 3-34, (1996).

[65] Roos, C., Terlaky, T. and Vial, J. *Interior point methods for linear optimization*. Springer, 2nd ed., (2005).

[66] Nocedal, J. and Wright, S. *Numerical optimization*. Springer Series in Operations Research and Financial Engineering, 2nd ed., (2006).

[67] Vanderbei, R. *LOQO: An interior point code for quadratic programming*. Optimization Methods and Software 12, pp. 451-484, (1999).

[68] Mizuno, S. *Infeasible-Interior-Point algorithms*. Interior Point Methods of Mathematical Programming, Applied Optimization Vol. 5, pp. 159-187, (1996).

[69] Benson, H., Shanno, D. and Vanderbei, R. *Interior point methods for nonconvex nonlinear programming: jamming and comparative numerical testing*. Operations Research and Financial Engineering. Princeton University, (2000).

[70] Frisch, K. *The logarithmic potential method of convex programming*. Technical report, University Institute of Economics, Oslo, Norway, (1955).

[71] Fiacco, A. and McCormick, G. *Nonlinear programming: Sequential unconstrained minimization techniques*. John Wiley and Sons, (1968).

[72] Gill, P., Murray, W., Saunders, M., Tomlin, J. and Wright, M. *On the projected Newton barrier methods for linear programming and an equivalence to Karmakar's projective method*. Math. Programming, 36, pp. 183-209, (1986).

[73] Forsgren, A., Gill, P. and Wright, M. *Interior methods for nonlinear optimization*. SIAM Review, Vol. 44, No. 4, pp. 525-597, (2002).

[74] Luenberger, D. and Ye, Y. *Linear and Nonlinear Programming*. Springer, 3rd ed., (2008).

[75] Vanderbei, R. *Linear programming: Foundations and extensions*. Springer. International Series in Operation Research and Management Science, 4th ed., (2014).

[76] Shanno, D., Breitfeld, M. and Simantiraki, E. *Implementing barrier methods for nonlinear programming*. Interior Point Methods of Mathematical Programming, Applied Optimization Vol. 5, pp. 399-414, (1996).

[77] Marsten, R., et. al. *Interior point methods for linear programming: Just call Newton, Lagrange and Fiacco and McCormick!* Interfaces, Vol. 20, No. 4, pp. 105-116, July-August (1990).

[78] Niu, L. and Yuan, Y. *A new trust-region algorithm for nonlinear constrained optimization*. Journal Of Computational Mathematics, Vol. 28, No. 1, pp. 72-86, (2010).

[79] Byrd, R., Hribar, M. and Nocedal, J. *An interior point algorithm for large scale nonlinear programming*. SIAM Journal On Optimization, 9(4):877-900, (1999).

[80] Ruszczynski, A. *Nonlinear Optimization*. Princeton University Press, (2006).

[81] Lusting, I., Marsten, R. and Shanno, D. *Interior Points methods for linear programming: Computational state of the art*. ORSA Journal on Computing. Vol. 6, No. 1, (1994).

[82] Lusting, I. J., Marsten, R. E. and Shanno, D. F. *Computational experience with a primal-dual interior point method for linear programming*. Elsevier Sciences Publishing, Linear Algebra And Its Applications, 152: 191-222, (1991).

[83] Monteiro, R. C. and Adler, I. *Interior path following primal-dual algorithms. Part I: Linear programming*. Math. Programming, 44: 27-42, (1989).

[84] Stott, B., Jardim, J. and Alsac, O. *DC power flow revisited*. IEEE Trans. Power Systems, Vol. 1290, No. 3, pp. 1290-1300, August (2009).

[85] Wu, F. *Theoretical study of the convergence of the of the fast decoupled load flow*. IEEE Trans. Power App. Syst., Vol. PAS-96, pp. 268-275, Jan/Feb (1977).

[86] Tripathy, S., Durga, G., Malik, O. and Hope, G. *Load-Flow solutions for ill-conditioned power systems by a Newton-like method*. IEEE Trans. Power Apparatus and Systems, Vol. PAS-101, No. 10, pp. 3648-3657, October (1982).

[87] Rajičić, D., Bose, A. *A modification to the fast decoupled power flows for networks with high r/x ratios*. IEEE Trans. Power Systems, Vol. 3, No. 2, pp. 743-746, May (1988).

[88] Wang, Y., Silva, L., Xu, W. and Zhang, Y. *Analysis of ill-conditioned power-flow problems using voltage stability methodology*. IEE Proc.-Gener. Transm. Distrib. Vol. 148, No. 5, pp. 384-390, September (2001).

[89] Milano, F. *Continuous Newton's method for power flow analysis*. IEEE Trans. on Power Systems, Vol. 24, No. 1, February, (2009).

[90] Gutiérrez, J., Bedriñana, M. and Castro, C. *Critical comparison of robust load flow methods for ill-conditioned systems*. IEEE Trondheim Power Tech 2011, paper ID: 546, 19-23 June, (2011).

[91] Grainger, J. and Stevenson, W. *Power system analysis*. McGraw-Hill International editions, (1994).

[92] Glover, J., Sarma, M and Overbye, T. *Power system analysis and design*. Cengage Learning, 5th ed., (2008).

[93] Romero, R., Monticelli, A., García, A. and Haffner, S. *Test systems and mathematical models for transmission network expansion planning*. IEE Proc.-Gener. Transm. Distrib. Vol. 149, No. 1, pp. 27-36, (2002).

[94] Fang, R. and Hill, D. *A new strategy for transmission expansion in competitive electricity markets*. IEEE Trans. on Power Systems, Vol 18, No. 1, pp. 374-380, February, (2003).

[95] Taha, H. *Integer programming: Theory, applications and computations*. Academic Press, (1975).

[96] Milano, F., Vanfretti, L. and Morataya, J. *An open source power system virtual laboratory: The PSAT case and experience*. IEEE Trans. on Education, Vol. 51, No. 1, February, (2008).

Appendix A

Fundamental Theorem of Linear Programming

A LP is a constrained optimization problem in which the objective function and each of the constrains are linear in the unknowns. Of course the set of constraints can include equality and/or inequality functions which defines the feasible solutions set (*feasible region*). However, for easy manipulation –which means adding or subtracting (nonnegative) slack variables in the inequality constraints– any LP can be transformed into the so-called *standard form*:

$$\begin{array}{rrcl} \min & c^T x & & \\ \text{subject to} & A\,x & = & b \\ & x & \geq & 0 \end{array} \tag{A.1}$$

where $x, c^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $m < n$, $rank\,A = m$ and $b \in \mathbb{R}^m$. The feasible region is:

$$\Omega = \{x \in R^n \,|\, Ax = b, x \geq 0\} \tag{A.2}$$

The main issue in this appendix is to show the *Fundamental Theorem of Linear Programming.* This important theorem is related to the next idea. As the feasible region (Ω) is a convex subset of $\mathbb{R}^n$ (which will be proven), we can minimize the objective function of (A.1) by following a descent direction which could be given by the negative gradient of $c^T x$, *i.e*, we could move in the direction of $-c$. It should be clear that the furthest feasible point in this direction would be lying on the boundary of the polyhedron formed by Ω. The Fundamental Theorem establishes not only this fact but also that the optimal solution will be in a vertex of the polyhedron.

A.1 Geometry of a LP

Let's begin the study of the geometry of LP with the definition of convex set and two theorems related to the feasible region. Then, we will give some useful definitions and assumptions for the *Fundamental Theorem.*

A.1.1 Convexity of the feasible region

Definition A.1 (Convex set.) *A set $Q \subset \mathbb{R}^n$ is called convex if for any $x, y \in Q$*

$$L = \{z \mid z = \alpha x + (1-\alpha)y,\ \alpha \in [0,1]\} \subseteq Q.$$

The feasible region Ω in (A.2) is defined by a set of equations in the form $A\,x = b$. Each of them is a hyper-plane which forms a convex set and we will prove this now.

Theorem A.1 *A hyper-plane is a convex set.*

Proof. Consider a hyper-plane in $\mathbb{R}^n$ defined by

$$a^T x = b \tag{A.3}$$

Let x, y be a couple of points that satisfies (A.3). We most show that $z = \alpha x + (1-\alpha)y$ forms a subset of the hyper-plane for all $\alpha \in [0,1]$, *i.e.* that $a^T z = b$.

We have:

$$a^T z = a^T\left[\alpha x + (1-\alpha)y\right] = \alpha a^T x + a^T y - \alpha a^T y$$

but x, y are two points in the hyper-plane, hence:

$$a^T z = \alpha b + b - \alpha b$$

therefore:

$$a^T z = b$$

□

Note that Ω is formed by the intersection of m hyper-planes. We will show that Ω is also a convex set.

Theorem A.2 *The finite intersection of convex sets is a convex set.*

Proof. Let $C_1, C_2, \ldots, C_m$ be convex sets, and

$$D = \bigcap_{i=1}^{m} C_i$$

Lets take $x, y \in D$, this implies that $x, y \in C_1, C_2, \ldots, C_m$. From hypothesis, every C_i, $i = 1, \ldots, m$ is convex. This means that

$$\alpha x + (1-\alpha)y \in C_1, C_2, \ldots, C_m \quad \forall \alpha \in [0,1]$$

as a consequence

$$\alpha x + (1-\alpha)y \in \bigcap_{i=1}^{m} C_i \quad \forall \alpha \in [0,1]$$

□

A.1.2 Basic solutions

A vector x satisfying $A\,x = b$ in (A.1) is called a *solution* to the LP. In addition, if $x \geq 0$, the vector is said to be a *feasible solution.* We will refer to a feasible solution that achieves the minimum value of the objective function as an *optimal feasible solution.*

In order to set the basis for the Fundamental Theorem, some necessary assumptions and definitions are given now.

Assumption A. The number of constraints are less than the number of variables. It is assumed that $m < n$, since if $m > n$, at least $m - n$ equations must be redundant, and $m = n$ results in a trivial situation in which $Ax = b$ has a unique solution point if the system is consistent.

Assumption B. Full rank assumption. The rank of A is m, this means that there are m linearly independent columns and rows of A. Particularly, the m equations on the constraints are linearly independent. A linear dependency among the rows of A would lead either to contradictory constraints and hence no solutions of (A.1), or to a redundancy that could be eliminated.

Lets consider a partition of the matrix A in such a way that $A = [B,\, N]$, where $B \in \mathbb{R}^{m \times m}$ is a nonsingular matrix formed by the first m linearly independent columns of A; in this case, B form a basis. Also, let $x = (x_B,\, x_N)$. Then $A\,x = b$ can be written as

$$[B,\, N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b \tag{A.4}$$

Since B is nonsingular, we can solve (A.4) for x_B to obtain:

$$x_B = B^{-1}\,(b - N x_N)$$

Definition A.2 (Basic Solution.) *The particular solution to* (A.4), $x_B = B^{-1}\,b$, *where* $x_N = 0$, *is called a* basic solution. *The elements of* x_B *are the* basic variables.

Definition A.3 (Basic Feasible Solution.) *If* x_B *is a basic solution and* $x_B \geq 0$, *we refer to it as a* basic feasible solution.

Observation A.1 (Geometrical equivalence of a basic feasible solution) *Geometrically, a* basic feasible solution *defines an* extreme point *(a vertex) of the solution space given by* Ω*; a proof of this is given in [59] and in [66].*

Observation A.2 (Degenerate basic solution) *In general, an extreme point is uniquely defined by a basic solution except when in an* $n-$*dimensional problem, there are more that* n *hyperplanes passing through the same vertex. In this case, the same extreme point* *is identified by more than one basic solution and the solution is said to be* degenerate*; however, it is impossible to identify more than one extreme point with the same basic solution.*

Finally, if the optimal feasible solution is basic, it is an *optimal basic feasible solution.*

A.2 Fundamental Theorem

We are ready to state the main theorem of this appendix.

Theorem A.3 (Fundamental Theorem of Linear Programming.) *Given a LP in standard form* (A.1) *where* $A \in \mathbb{R}^{m \times n}$ *of rank* m*:*

i. if there is a feasible solution, there is a basic feasible solution;

ii. if there is an optimal feasible solution, there is an optimal basic feasible solution

Proof. See [74], section 2.4, pp. 20. □

Observation A.3 *The Fundamental Theorem gives an strategy to "reduce" the search of the optimal solution from the whole feasible space (infinite points) to the search of "only" the basic solutions (finite points). Therefore, applying a "brute-force approach" we can solve a LP by comparing all basic solutions (perhaps choosing first the basic* feasible *solutions) and then the one that minimizes the objective function. This approach depends on the number of basic solutions which, however, is a finite number, is bounded by:*

$$C_m^n = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

which is in general a very large number.

For instance, we have:

- $C_{34}^{46} = 3.8910 \times 10^{10}$ basic solutions for the Garver's formulation WOR, and
- $C_{36}^{49} = 2.6259 \times 10^{11}$ basic solutions for the Garver's formulation WR.

Of course this approach is not practical and a more efficient method for solving this problems is needed. Just for example, supposing that we try to use this method and we have a computer that takes 1 μsec (10^{-6} seconds) to test every basic solution. The computer would need 10.8 hours to solve every LP resulting in the formulation WOR for the Garver's system in the worst case; considering the 7 iterations of the CHA process this means 75.6 hours for solving completely the STEP problem). For the WR formulation it would take 72.9 hours for every LP and 291.6 hours for the complete STEP problem.

Appendix B

The Infeasible IPM for LP: A two variables example

We will consider in this appendix the following general LP problem

$$\begin{array}{rlll} \min & c^T x & & \\ \text{subject to} & A\,x & = & b \\ & B\,x & \geq & d \end{array} \tag{B.1}$$

where $c, x \in \mathbb{R}^n$, $A \in \mathcal{M}_{p\times q}$, $b \in \mathbb{R}^p$, $B \in \mathcal{M}_{q\times n}$ and $d \in \mathbb{R}^q$.

In the form of (3.3) we have:

$$\begin{array}{rlll} \min & c^T x & & \\ \text{subject to} & A\,x - b & = & 0 \\ & B\,x - d & \geq & 0 \end{array} \tag{B.2}$$

and it is clear that, in this case $f(x) = c^T x$, $g(x) = A\,x - b$ and $h(x) = B\,x - d$.

Working as in section 3.2 we can write the Lagrangian as follows:

$$\mathcal{L}_\mu(w; \mu^k) = c^T\,x - \mu^k \sum_{i=1}^{q} log(s_i) - \lambda^T\,[A\,x - b] - \pi^T\,[B\,x - d - s] \tag{B.3}$$

Here, $g^T(x) = (A\,x - b)^T = (A\,x)^T - b^T = x^T A^T - b^T$. Thus, for the perturbed primal-dual system $\nabla_x g^T(x) = \nabla_x(x^T A^T - b^T) = A^T$.

By proceeding analogously, the perturbed primal-dual system is:

$$\begin{array}{rcrcl} \nabla_x \mathcal{L} & = & c - A^T\,\lambda - B^T\,\pi & = & 0 \\ \nabla_s \mathcal{L} & = & -\mu^k e + S\,\Pi\,e & = & 0 \\ \nabla_\lambda \mathcal{L} & = & -A\,x + b & = & 0 \\ \nabla_\pi \mathcal{L} & = & -B\,x + d + s & = & 0 \end{array} \tag{B.4}$$

The Hessian for the Lagrangian (B.3) is:

$$\nabla^2 \mathcal{L}_\mu(w) = \begin{bmatrix} 0 & 0 & -A^T & -B^T \\ 0 & \Pi & 0 & S \\ -A & 0 & 0 & 0 \\ -B & I & 0 & 0 \end{bmatrix} \tag{B.5}$$

Thus we can write

$$\begin{bmatrix} 0 & 0 & -A^T & -B^T \\ 0 & \Pi & 0 & S \\ -A & 0 & 0 & 0 \\ -B & I & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta s \\ \Delta\lambda \\ \Delta\pi \end{bmatrix} = \begin{bmatrix} -c + A^T\,\lambda + B^T\,\pi \\ \mu^k e - S\,\Pi\,e \\ A\,x - b \\ B\,x - d - s \end{bmatrix} \tag{B.6}$$

which will be used to compute the search direction.

As an example, we will develop one iteration of the IPM process of a two variables linear program with a solution $x_1 = 5.0$ and $x_2 = 7.5$, and with an objective function value of 165.0.

Example B.1 *Consider the following LP:*

$$\begin{array}{lrcr} \textit{min} & z = 15\,x_1 + 12\,x_2 & & \\ \textit{subject to} & x_1 + 2\,x_2 & \geq & 20 \\ & 3\,x_1 + 2\,x_2 & \geq & 30 \\ & x_1 & \geq & 0 \\ & x_2 & \geq & 0 \end{array} \tag{B.7}$$

In this example, $w \in \mathbb{R}^{10}$, more specific:

$$w^T = [x \;\; s \;\; \pi] = [x_1\,x_2 \;\; s_1\,s_2\,s_3\,s_4 \;\; \pi_1\,\pi_2\,\pi_3\,\pi_4]$$

There are not λ multipliers because the problem does not have any equality constraint.

Taking advantage of the geometry visualization of the feasible region (Figure B.1), we will take as the initial point, the vector:

$$(w^0)^T = \begin{bmatrix} x^0 & s^0 & \pi^0 \end{bmatrix} = [10\,10 \;\; 1\,1\,1\,1 \;\; 1\,1\,1\,1]$$

where s^0 and π^0 are strictly positive.

Using these vectors, the complementarity gap (3.24) is: $\rho^0 = (s^0)^T\,\pi^0 = 4$. Considering the equation (3.26) and a centering parameter $\sigma = 0.1$, the initial barrier parameter is $\mu^0 = 0.1$.

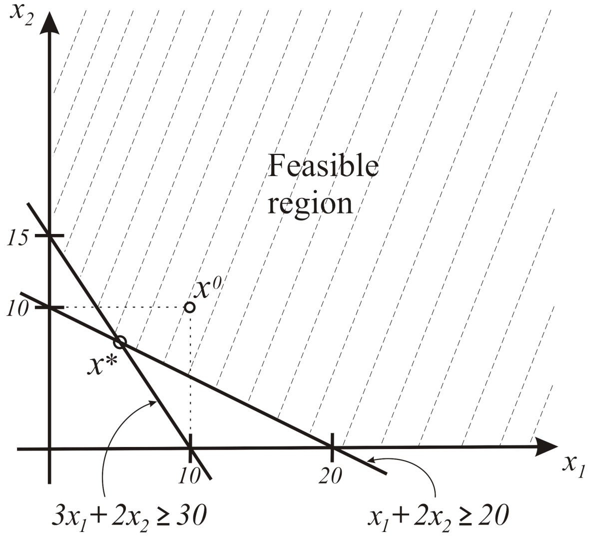

Figure B.1: Feasible region of the two variables example.

Including the non-negativity conditions on the inequality constraints, the formulation will have the following vectors and matrices:

$$c^T = \begin{bmatrix} 15 & 12 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad d = \begin{bmatrix} 20 \\ 30 \\ 0 \\ 0 \end{bmatrix}$$

According to the Chapter 3, subsection 3.2.1, we have:

$$S = \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & s_4 \end{bmatrix} \quad \Rightarrow \quad S^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Analogously

$$\Pi = \begin{bmatrix} \pi_1 & 0 & 0 & 0 \\ 0 & \pi_2 & 0 & 0 \\ 0 & 0 & \pi_3 & 0 \\ 0 & 0 & 0 & \pi_4 \end{bmatrix} \quad \Rightarrow \quad \Pi^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the Hessian (B.5) *for the iteration zero is:*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rows of the right-hand side of (B.6) *are:*

$$-c + B^T \pi^0 = -\begin{bmatrix} 15 \\ 12 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -7 \end{bmatrix}$$

$$\mu^0 e - S\,\Pi\, e = 0.1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.9 \\ -0.9 \\ -0.9 \\ -0.9 \end{bmatrix}$$

$$B\,x^0 - d - s^0 = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 20 \\ 30 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 19 \\ 9 \\ 9 \end{bmatrix}$$

Thus, the equation (B.6) *for the search directions can be written now as follows:*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta s_1 \\ \Delta s_2 \\ \Delta s_3 \\ \Delta s_4 \\ \Delta \pi_1 \\ \Delta \pi_2 \\ \Delta \pi_3 \\ \Delta \pi_4 \end{bmatrix} = \begin{bmatrix} -10 \\ -7 \\ -0.9 \\ -0.9 \\ -0.9 \\ -0.9 \\ 9 \\ 19 \\ 9 \\ 9 \end{bmatrix}$$

Solving the last expression we have:

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta s_1 \\ \Delta s_2 \\ \Delta s_3 \\ \Delta s_4 \\ \Delta \pi_1 \\ \Delta \pi_2 \\ \Delta \pi_3 \\ \Delta \pi_4 \end{bmatrix} = \begin{bmatrix} -5.53 \\ -3.59 \\ -3.70 \\ -4.76 \\ 3.47 \\ 5.41 \\ 2.80 \\ 3.86 \\ -4.37 \\ -6.31 \end{bmatrix}$$

Using these values, we can compute:

$$\begin{aligned} max\left\{-\frac{\Delta s_i^0}{s_i^0}\right\} &= max\{3.70 \quad 4.76 \quad -3.47 \quad -5.41\} = 4.76 \\ max\left\{-\frac{\Delta \pi_i^0}{\pi_i^0}\right\} &= max\{-2.80 \quad -3.86 \quad 4.37 \quad 6.31\} = 6.31 \end{aligned}$$

Now, using (3.20) *and* $\alpha_0 = 0.95$*, the primal and dual step lengths are:*

$$\begin{aligned} \alpha_p^0 &= 0.1996 \\ \alpha_d^0 &= 0.1506 \end{aligned}$$

The variable updating (3.21) *and* (3.22) *are:*

$$\begin{aligned} x_1^1 &= x_1^0 + \alpha_p^0 \Delta x_1^0 &= 8.90 \\ x_2^1 &= x_2^0 + \alpha_p^0 \Delta x_2^0 &= 9.28 \end{aligned}$$

$$\begin{aligned} s_1^1 &= s_1^0 + \alpha_p^0 \Delta s_1^0 &= 0.26 \\ s_2^1 &= s_2^0 + \alpha_p^0 \Delta s_2^0 &= 0.05 \\ s_3^1 &= s_3^0 + \alpha_p^0 \Delta s_3^0 &= 1.69 \\ s_4^1 &= s_4^0 + \alpha_p^0 \Delta s_4^0 &= 2.08 \end{aligned}$$

$$\begin{aligned} \pi_1^1 &= \pi_1^0 + \alpha_d^0 \Delta \pi_1^0 &= 1.42 \\ \pi_2^1 &= \pi_2^0 + \alpha_d^0 \Delta \pi_2^0 &= 1.58 \\ \pi_3^1 &= \pi_3^0 + \alpha_d^0 \Delta \pi_3^0 &= 0.34 \\ \pi_4^1 &= \pi_4^0 + \alpha_d^0 \Delta \pi_4^0 &= 0.05 \end{aligned}$$

This and the following iterations are shown in table B.1. The tolerances used for this example are: $\varepsilon_1 = 10^{-2}$, $\varepsilon_2 = 10^{-3}$ and $\varepsilon_\mu = 10^{-6}$.

Table B.1: Results for the numerical example

k	μ^k	x^k		s^k				π^k				z
		x_1	x_2	s_1	s_2	s_3	s_4	π_1	π_2	π_3	π_4	
1	0.1	8.8959	9.2839	0.2611	0.0500	1.6932	2.0812	1.4213	1.5803	0.3423	0.0500	244.8464
2	0.0283	7.4875	8.6699	0.1853	0.0025	2.8455	4.0279	1.3783	2.4881	0.0649	0.0025	216.3513
3	0.0114	5.1897	7.5836	0.0093	0.0025	4.8421	7.2360	1.4335	3.6273	0.0032	0.0015	168.8489
4	0.0012	5.0267	7.5122	0.0017	0.0001	4.9772	7.4627	1.4997	4.4957	0.0002	0.0001	165.5475
5	1.2×10^{-4}	5.0000	7.5001	0.0001	0.0000	4.9999	7.5000	1.5000	4.5004	0.0000	0.0000	165.0017
6	8.7×10^{-6}	5.0000	7.5000	0.0000	0.0000	5.0000	7.5000	1.5000	4.5000	0.0000	0.0000	165.0000

Appendix C

Power Flow Results for the Garver's Test System

The PSAT (Power System Analysis Toolbox) is an open source Matlab toolbox for electric power system analysis and simulation capable of solving studies as power flow, continuation power flow and/or voltage stability analysis, the optimal power flow, the small-signal stability analysis and the time-domain simulation [58]. As a free power system software, its use with educational and research purposes has been extended to several universities as it is reported in [96] and the references therein.

In this work, we use the PSAT in order to check the steady state conditions before and after the expansion of the Garver's test system. The output of the PSAT for the power flow analysis can be displayed in graphical form (2D or 3D) or in a static report. The color bar in the right of the graphical output shows the values of the parameter studied (power flows, voltages, etc.) in p.u.

C.1 Initial condition

The Garver's system begins with a power network working within its normal operating parameters, giving excellent power flow measures as it can be seen in Figure C.1 and in the 3D perspective shown in Figure C.2.

In Table C.1 we show a summary of the static report given by the PSAT, where the bus voltages profiles and the real power losses are considered.

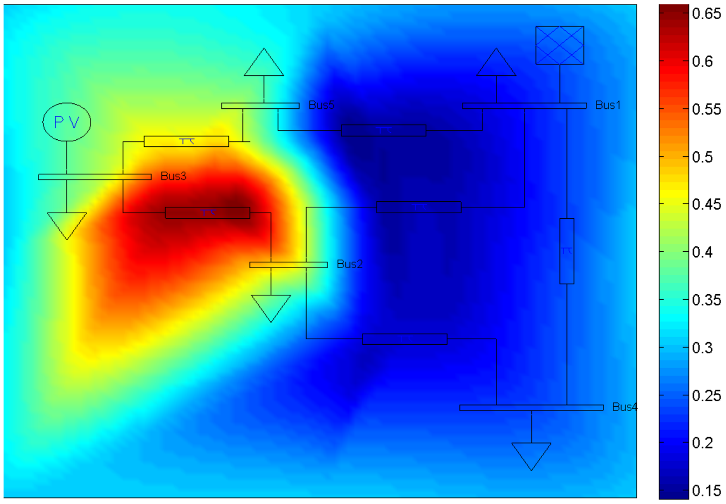

Figure C.1: 2D Power flow scenario for the initial configuration.

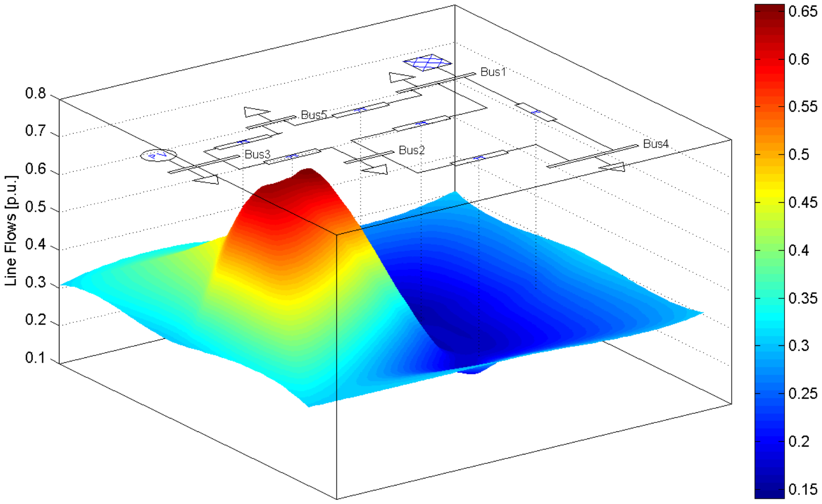

Figure C.2: 3D Power flow scenario for the initial configuration.

Table C.1: PSAT Power flow report for the initial condition

Power Flow Results					**Line Flows**			
Bus	V	phase	P_G	P_D	ROW	P Flow	ROW	P Flow
	[$p.u.$]	[deg]	[$p.u.$]	[$p.u.$]		[$p.u.$]		[$p.u.$]
1	1	0	0.7472	0.2	1-2	0.1645	2-1	-0.1616
2	0.9665	-3.6219	0	0.6	1-4	0.2328	4-1	-0.2244
3	1	3.8182	1.2	0.1	1-5	0.1499	5-1	-0.1486
4	0.9491	-8.0746	0	0.4	2-3	-0.6175	3-2	0.6380
5	0.9824	-1.5957	0	0.6	2-4	0.1791	4-2	-0.1756
Total			1.9472	1.9	3-5	0.4621	5-3	-0.4514
Real power losses			0.04717					

It is clear that the system operates properly under these initial conditions. However, as it was aforementioned in subsection 4.5.1, a future condition for the Garver's system is expected and it will be necessary to know how the system will operate. This is shown below.

C.2 Future condition

Under the new conditions given in subsection 4.5.1, the power flow study gives the results showed in Figures C.3 and C.4, and in Table C.2.

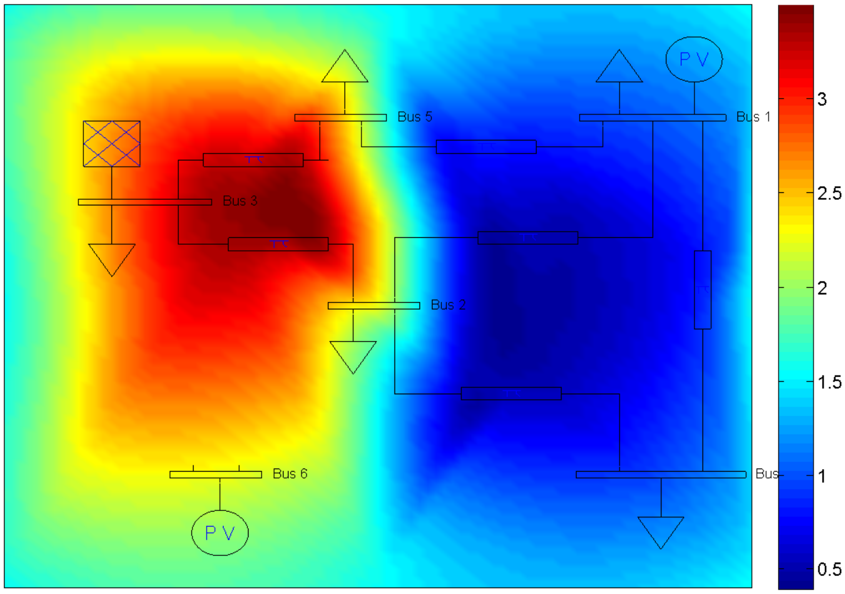

Figure C.3: 2D Power flow scenario for the future condition.

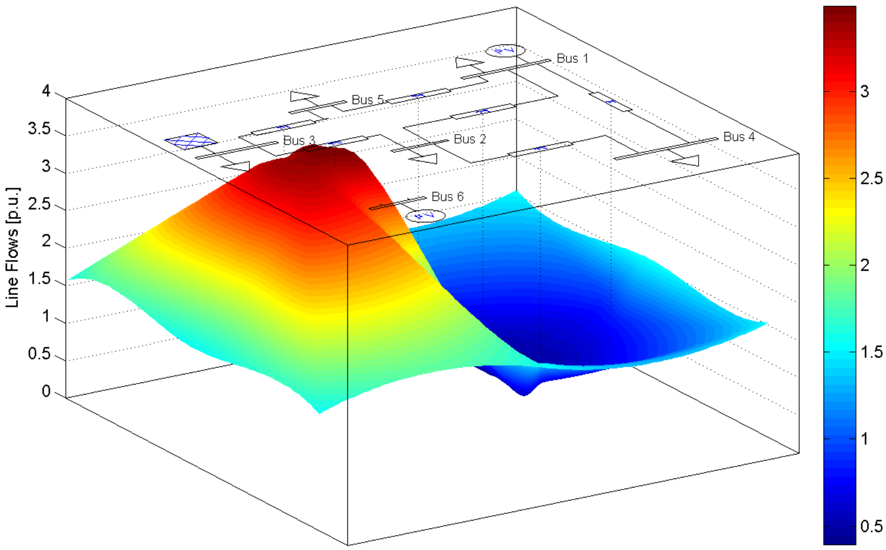

Figure C.4: 3D Power flow scenario for the future condition.

Table C.2: PSAT Power flow report for the future condition

	Power Flow Results					**Line Flows**		
Bus	V	phase	P_G	P_D	ROW	P Flow	ROW	P Flow
	$[p.u.]$	$[deg]$	$[p.u.]$	$[p.u.]$		$[p.u.]$		$[p.u.]$
1	1	-45.9055	1.5	0.8	1-2	0.1145	2-1	-0.0557
2	0.6859	-43.3971	0	1.7643	1-4	0.6068	4-1	-0.5011
3	1	0	6.1887	0.4	1-5	-0.0213	5-1	0.0747
4	0.6256	-72.1935	0	0.9784	2-3	-2.2505	3-2	2.8078
5	0.7965	-41.8802	0	2.3788	2-4	0.5419	4-2	-0.4773
6	0	0	0	0	3-5	2.9809	5-3	-2.4534
	Total		7.6887	6.3215				
	Real power losses		1.3672					

Considering the PSAT power flow report (Table C.2), one could think about a reinforcement (another) on generation in bus 3, where the maximum capacity was exceeded, in order to avoid the generation expansion in bus 6 and the transmission expansion to connect that bus, hooping to save their related costs. However, the problem is not only connected to the power balance in the system but other complications arising from the new configuration. For example, there is no way to transport all the power generated in such bus to another buses where is needed, the ROWs 2-3 and 3-5 are overloaded (the maximum limit power flow in each of those corridors is 100 MW and the results show that there is a transfer of almost 300 MW); the forced load shedding in bus 2, 3 and 5, and the very low voltages profiles and the critical phase angles values are of interest as well.

C.3 Study WOR

The first proposal of expansion for the Garver's system was obtained according to the subsection 5.2.1 using a without redispatch modeling (where 7 new circuits in the system are needed). Figures C.5 and C.6 give the power flow behavior for this new configuration. The Table C.3 shows the power flow report.

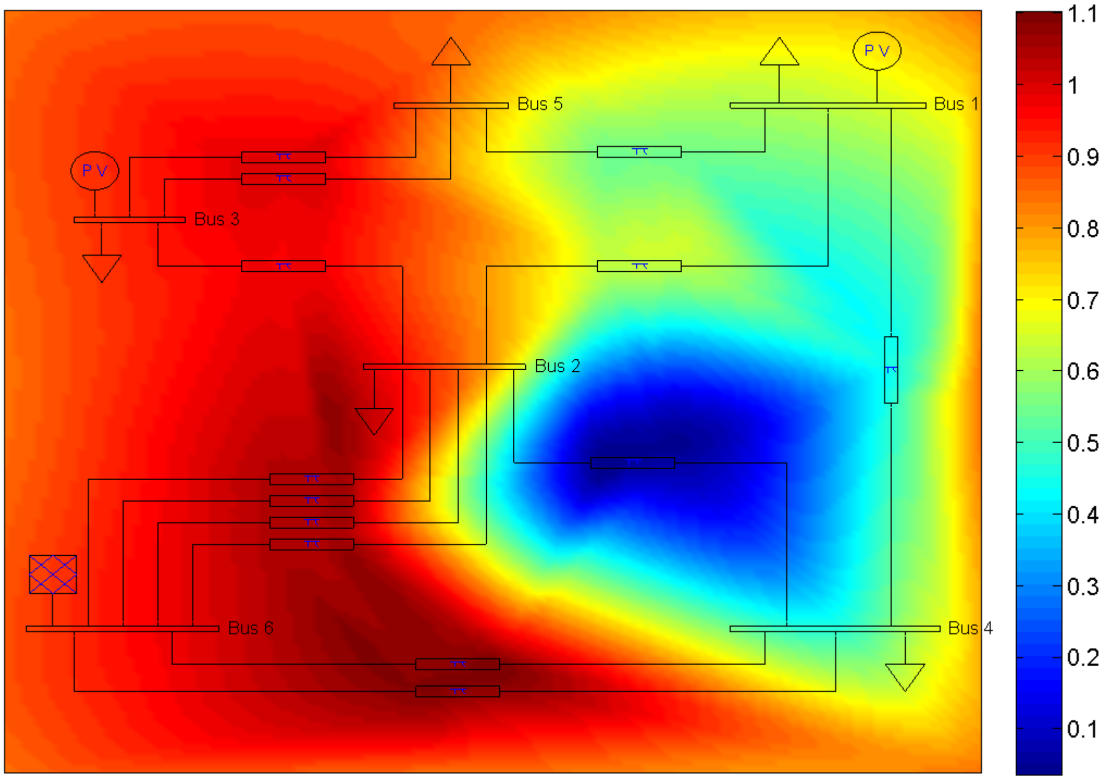

Figure C.5: 2D Power flow scenario for the WOR solution.

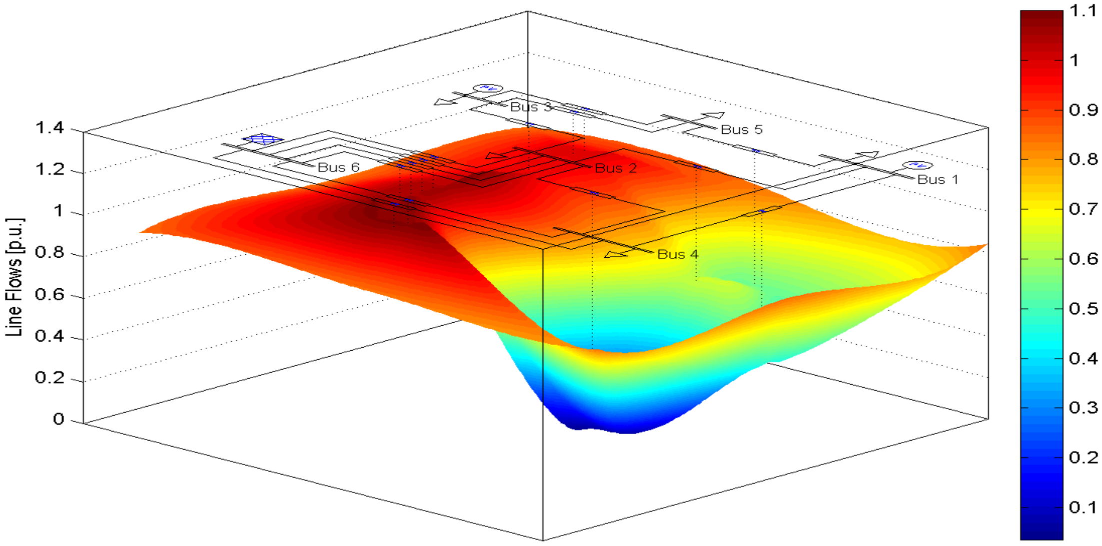

Figure C.6: 3D Power flow scenario for the WOR solution.

Table C.3: PSAT Power flow report expansion plan WOR

	Power Flow Results					**Line Flows**		
Bus	V	phase	P_G	P_D	ROW	P Flow	ROW	P Flow
	$[p.u.]$	$[deg]$	$[p.u.]$	$[p.u.]$		$[p.u.]$		$[p.u.]$
1	1	-34.9669	0.5	0.8	1-2	-0.5347	2-1	0.5815
2	0.9196	-18.7733	0	2.4	1-4	-0.3244	4-1	0.3544
3	1	-29.6937	1.65	0.4	1-5	0.5591	5-1	-0.5419
4	0.8966	-19.3833	0	1.6	2-3	0.7525	3-2	-0.7057
5	0.9422	-41.2391	0	2.4	2-4	0.0332	4-2	-0.0328
6	1	0	6.2214	0	3-5	0.9779 × 2	5-3	-0.9291 × 2
		Total	8.3714	7.6	2-6	-0.9418 × 4	6-2	1.028 × 4
		Real power losses	0.7714		4-6	-0.9608 × 2	6-4	1.054 × 2

In Table C.3 we can see that even when the expansion proposal without redispatch could be enough for a lossless study, some overloads could be observed when losses are considered (P Flow 6-2 and 6-4).

C.4 Study WR

The other proposal for the Garver's system expansion is through a modeling with redispatch where the optimal expansion given in subsection 5.2.2 indicates the necessity of 4 new circuits. The power flow study for this configuration is given in Figures C.7 and C.8 and in Table C.4.

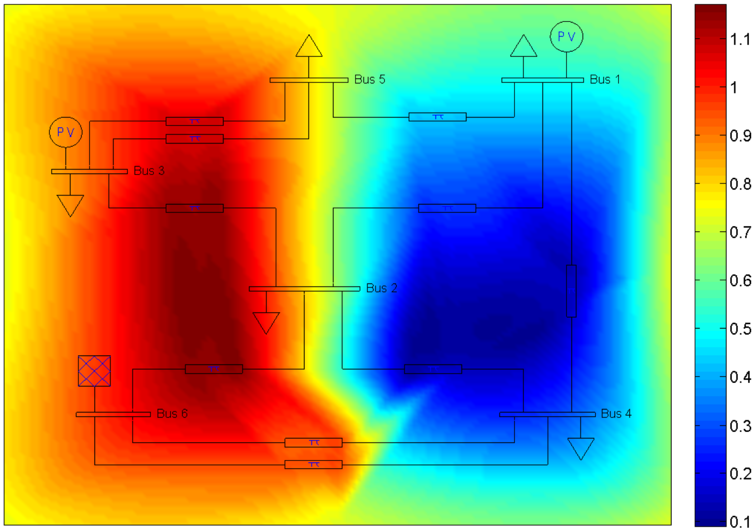

Figure C.7: 2D Power flow scenario for the WR solution.

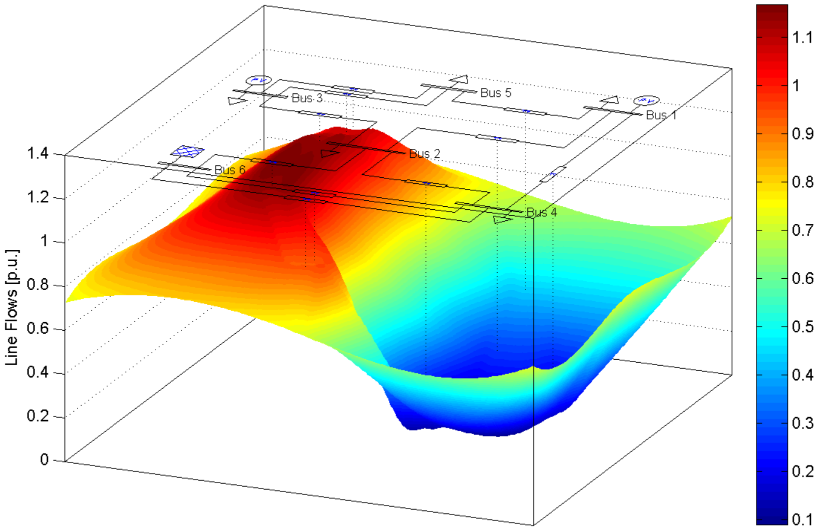

Figure C.8: 3D Power flow scenario for the WR solution.

Table C.4: PSAT Power flow report expansion plan WR

	Power Flow Results				**Line Flows**			
Bus	V $[p.u.]$	phase $[deg]$	P_G $[p.u.]$	P_D $[p.u.]$	ROW	P Flow $[p.u.]$	ROW	P Flow $[p.u.]$
1	1	-15.8507	1.5	0.8	1-2	0.2340	2-1	-0.2251
2	0.9054	-20.6033	0	2.4	1-4	0.0538	4-1	-0.0505
3	1	-7.7217	3.6	0.4	1-5	0.4122	1-5	0.4122
4	0.9088	-16.5749	0	1.6	2-3	-1.0238	3-2	1.0880
5	0.9407	-20.2454	0	2.4	2-4	-0.1366	4-2	0.1390
6	1	0	2.9457	0	3-5	1.056 × 2	5-3	-0.9992 × 2
		Total	8.0457	7.6	2-6	-1.0145	6-2	1.1181
		Real power losses	0.4457		4-6	-0.8442 × 2	6-4	0.9138 × 2

Appendix D

Data of the Test Systems

In this appendix, we summarize the data for the generation-demand and for the right of ways of the test systems studied in this work. The initial topology configuration of each system is included as well.

D.1 6-bus Garver Test System [40]

Table D.1: Generation and load data for Garver's system

Bus No.	$\bar{g}_i$ $[MW]$	d_i $[MW]$
1	150	80
2	—	240
3	360	40
4	—	160
5	—	240
6	600	—
Total	1110	760

Table D.2: Right of way data for Garver's system

Circuit	n_{ik}^0	**Cost,** $\times 10^3$ $[USD]$	$\bar{f}_{ik}$ $[MW]$
1-2	1	40	100
1-4	1	60	80
1-5	1	20	100
2-3	1	20	100
2-4	1	40	100
3-5	1	20	100
2-6	0	30	100
4-6	0	30	100

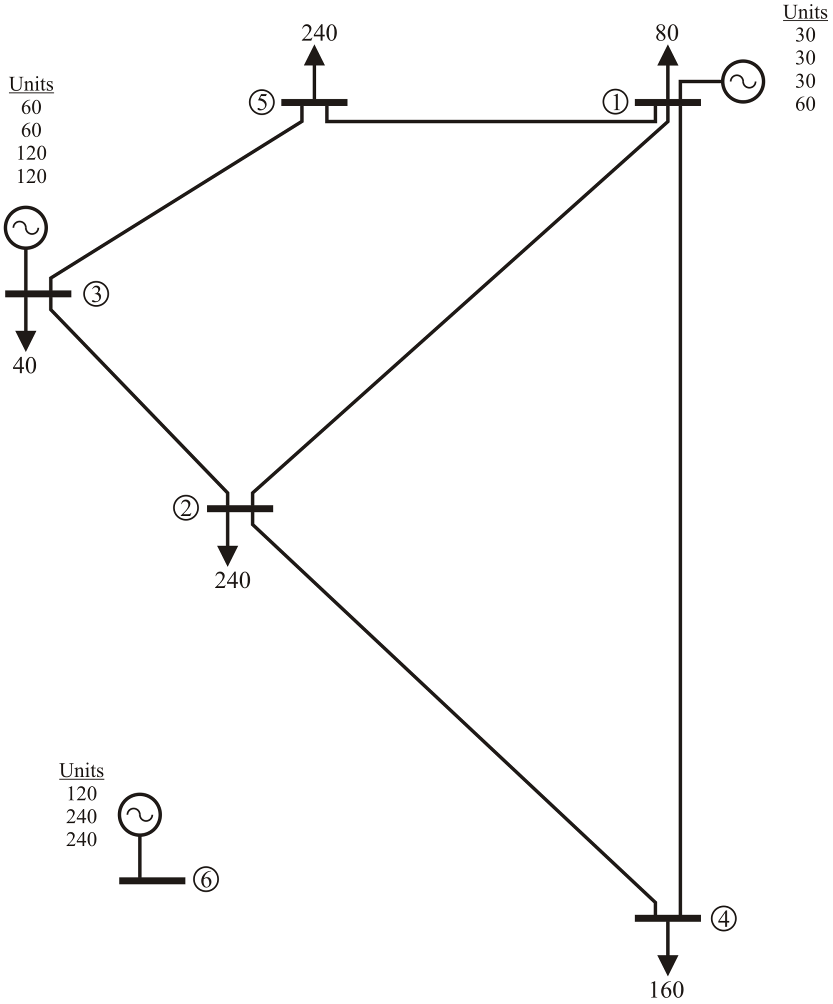

Figure D.1: Future condition –initial topology for the STEP modeling– of Garver's system.

D.2 24-bus IEEE Test System [94]

Table D.3: Generation and load data for the 24-bus IEEE test system

Bus No.	$\bar{g}_i$ $[MW]$	d_i $[MW]$
1	576	324
2	576	291
3	—	540
4	—	222
5	—	213
6	—	408
7	900	375
8	—	513
9	—	525
10	—	585
11	—	—
12	—	—
13	1773	795
14	—	582
15	645	951
16	465	300
17	—	—
18	1200	999
19	—	543
20	—	384
21	1200	—
22	900	—
23	1980	—
24	—	—
Total	10215	8550

Table D.4: Right of way data for the 24-bus IEEE test system

Right of way	n_{ik}^0	**Cost,** 10^4 $[USD]$	$\overline{f}_{ik}$ $[MW]$
1-2	1	3	175
1-3	1	55	175
1-5	1	22	175
2-4	1	33	175
2-6	1	50	175
3-9	1	31	175
3-24	1	50	400
4-9	1	27	175
5-10	1	23	175
6-10	1	16	175
7-8	1	16	175
8-9	1	43	175
8-10	1	43	175
9-11	1	50	400
9-12	1	50	400
10-11	1	50	400
10-12	1	50	400
11-13	1	66	500
11-14	1	58	500
12-13	1	66	500
12-23	1	134	500
13-23	1	120	500
14-16	1	54	500
15-16	1	24	500
15-21	2	68	500
15-24	1	72	500
16-17	1	36	500
16-19	1	32	500
17-18	1	20	500
17-22	1	146	500
18-21	2	36	500
19-20	2	55	500
20-23	2	30	500
21-22	1	94	500
1-8	0	35	500
2-8	0	33	500
6-7	0	50	500
13-14	0	62	500
14-23	0	86	500
16-23	0	114	500
19-23	0	84	500

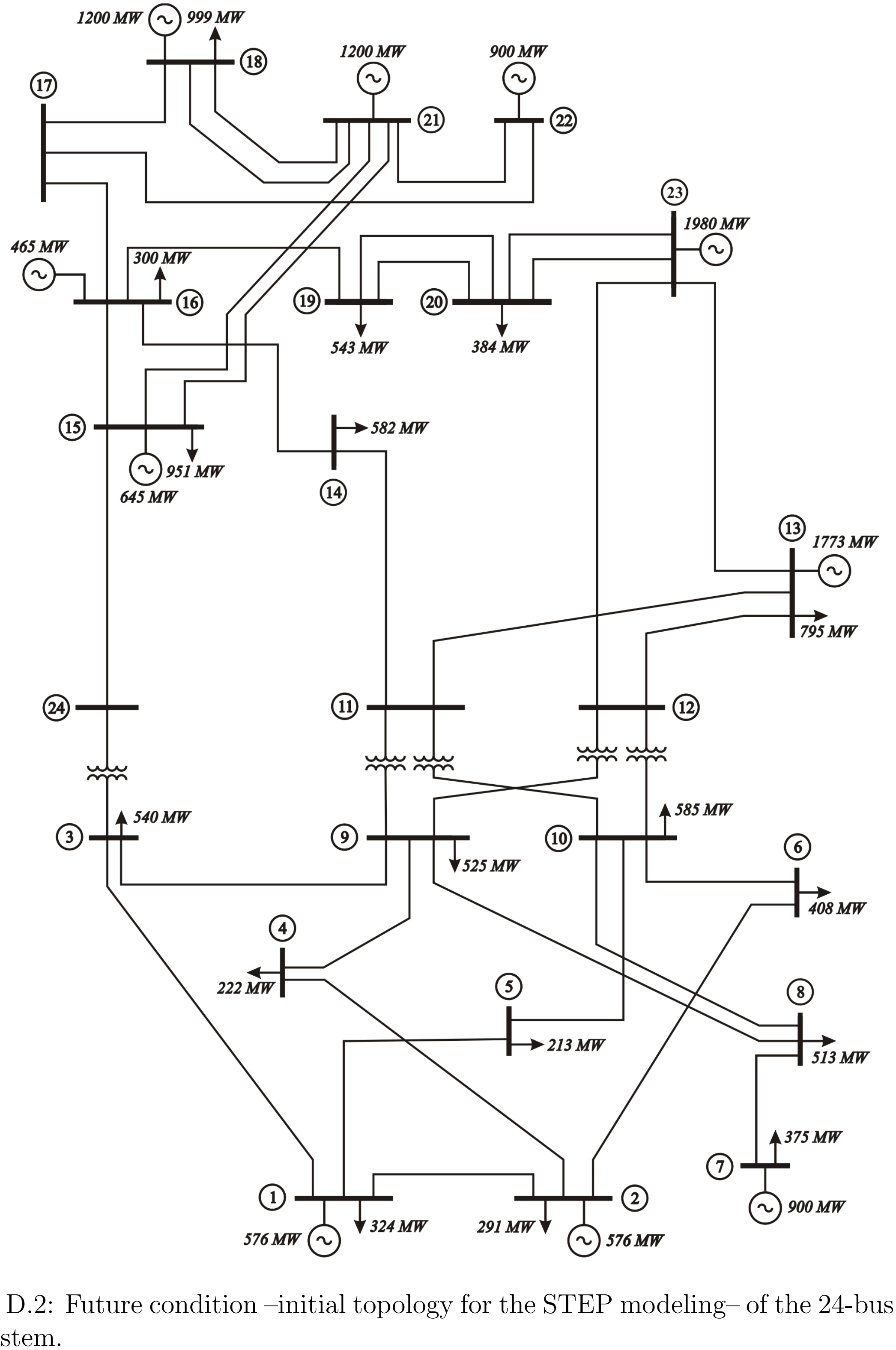

Figure D.2: Future condition –initial topology for the STEP modeling– of the 24-bus IEEE test system.