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# "Higher Topological Complexity in Real Flag Manifolds"

## TESIS

QUE PARA OBTENER EL GRADO DE:

**Doctor en Ciencias Fisicomatemáticas**

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# Resumen


En este trabajo de tesis se abordan dos problemas relacionados con el concepto de Complejidad Topológica (TC por sus siglas en inglés) el cual es un invariante algebraico relacionado con el problema de planeación motriz en robótica. $\mathrm{TC}(X)$ mide la discontinuidad de los algoritmos de planeación motriz en un espacio de estados $X$. Hallar este invariante requiere de la construcción explícita de planes motrices óptimos, por lo que hallar su valor de manera directa puede complicarse. En este trabajo se usan métodos indirectos los cuales acotan su valor; arriba por la dimensión y por abajo por medio de cálculos cohomológicos del espacio $X$. Este invariante se puede generalizar a un nuevo invariante $\mathrm{TC}_s$ que para el caso $s = 2$, coincide con TC.

En la primera parte de la tesis se trabaja con el problema de hallar $\mathrm{TC}(F(1^k, m))$ donde $F(1^k, m)$ es una variedad de banderas semi-completa, las cuales son generalizaciones del espacio proyectivo y de las grassmanianas; además $F(1^k, m)$ modela ciertos espacios de configuraciones.

En la segunda parte de la tesis se desarrolla una generalización del Teorema de Farber, Yuzvinsky y Tabachnikov el cual demuestra que, para casi todo $n$, $\mathrm{TC}(\mathbb{R}\mathrm{P}^n) = \mathrm{Imm}(\mathbb{R}\mathrm{P}^n)$, donde la segunda parte de la ecuación es la dimensión de inmersión suave de $\mathbb{R}\mathrm{P}^n$, cuyo cálculo explícito es un problema abierto en matemáticas. La generalización que buscamos es para $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^n)$ con $s \geq 3$.

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# Abstract


This thesis deals with two problems related to the concept of Topological Complexity (TC) which is an algebraic invariant connected with the motion planning problem in robotics. $\mathrm{TC}(X)$ parameter measures the order of discontinuity associated to the algorithms in motion planning of a state space $X$. Determining this invariant requires explicit optimal constructions of motion plannings therefore, to obtain its value in a direct way may be complicated. In this work we use indirect methods to bound the value of $\mathrm{TC}(X)$; upper bounds were obtained by the dimension and lower bounds by means of cohomological calculations of the space $X$. $\mathrm{TC}(X)$ invariant can be generalized as a new invariant, $\mathrm{TC}_s$, so that in the case $s = 2$, $\mathrm{TC}_2$ is equal to TC.

First part of this thesis concerns with the problem of finding $\mathrm{TC}(F(1^k, m))$, where $F(1^k, m)$ is a semicomplete real flag manifold, which generalizes projective space and the Grassmannians; besides $F(1^k, m)$ models certain configuration spaces.

Second part of this thesis carries out a generalization of Farber, Yuzvinsky and Tabachnikov's Theorem, according to which $\mathrm{TC}(\mathbb{R}\mathrm{P}^n) = \mathrm{Imm}(\mathbb{R}\mathrm{P}^n)$ for almost all $n$, where $\mathrm{Imm}(\mathbb{R}\mathrm{P}^n)$ is the smooth immersion dimension of $\mathbb{R}\mathrm{P}^n$, which its explicit calculation is an open problem in mathematics. We look for the generalization for $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^n)$ with $s \geq 3$.

# Agradecimientos

En esta parte del trabajo quiero hacer patente mi agradecimiento a las personas y circunstancias para que se llevara con éxito esta tesis doctoral. Estoy muy contento de poder dar este reconocimiento a todos aquellos que me ayudaron a culminar mi tesis el cual es un objetivo que he perseguido durante mucho tiempo.

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# Contents

# Chapter 1

# Introduction

The topological complexity (TC) of a space $X$ was introduced early this millennium by Michael Farber, as a way to use techniques from homotopy theory to study and model, from a topological perspective, the motion planning problem in robotics. If $P(X)$ stands for the space of free paths in $X$, then $\mathrm{TC}(X)$ is the reduced Schwarz genus (also known as sectional category) of the fibration $e\colon P(X) \to X \times X$ given by $e(\gamma) = (\gamma(0), \gamma(1))$. We refer the reader to the Section 2.1 and the references there in for a discussion of the meaning, relevance, and basic properties of Farber's concept.

The concept of topological complexity was generalized a few years later by Yuli Rudyak, who defined in [31] the $s$-th topological complexity of $X$, $\mathrm{TC}_s(X)$, as the reduced Schwarz genus, of fibration $e_s : P(X) \to X \times \cdots \times X = X^s$ given by evaluation of a path $\gamma : [0,1] \to X$ at $s$ given points of $[0,1]$ as we will see in detail in Section 2.2. In particular $\mathrm{TC} = \mathrm{TC}_2$. Rudyak's "higher" topological complexity has been studied systematically in [2].

The motion planning problem in robotics is a problem of current relevance due to the need to build robust algorithms in order to indicate different movements of the robots. The motion planning considered in this work addresses the issue of producing motion algorithms that are robust to noise.

The purposes of this thesis are manifold. In one hand, we give extensive computations to estimate the value of $\mathrm{TC}_s$ on a number of infinite families of semicomplete real flag manifolds $\mathbb{F}(1^k, m)$—the Grassmann type manifolds consisting of $(k+1)$-tuples $(L_1, \ldots L_k, V)$ of mutually orthogonal linear subspaces of $\mathbb{R}^{m+k}$ with $\dim(V) = m$ and $\dim(L_i) = 1$ for $1 \leq i \leq k$. Much of the motivation here comes from an unexpected connection between Farber's TC and one of the central problems in differential topology, namely the Euclidean Immersion Dimension for smooth manifolds. Explicitly, for a manifold $M$, let $\mathrm{Imm}(M)$ denote the dimension of the smallest Euclidean space where $M$ can be immersed. Then the main result in [15] asserts that TC = Imm for all real projective spaces $\mathbb{R}\mathrm{P}^m$ except for the only three parallelizable manifolds, $\mathbb{R}\mathrm{P}^1$, $\mathbb{R}\mathrm{P}^3$, and $\mathbb{R}\mathrm{P}^7$, for which the relation TC = Imm $-\,1$ holds. Of course, flag manifolds $\mathbb{F}(1^k, m)$ are a natural generalization of real projective spaces. So it is natural to ask whether the above relationship between topological complexity and immersion dimension also holds for the larger family of manifolds. Although the Euclidean immersion dimension of real flag manifolds is a much studied problem, and quite a lot of numeric information on it is available to date, the topological complexity of real flag manifolds had not been considered before—except, of course, for the already noted results with real projective spaces. The numeric TC-results obtained in this work show that the nice relationship between TC and Imm holding for real projective spaces $\mathbb{F}(1, m)$ *does not* hold for flag manifolds $\mathbb{F}(1^k, m)$ with $k > 1$. For instance, $\mathbb{F}(1, 1, 1)$ is a closed parallelizable 3-manifold (see [25]), so that $\mathrm{Imm}(\mathbb{F}(1, 1, 1)) = 4$. However, Theorem 1.1 below gives $\mathrm{TC}(\mathbb{F}(1, 1, 1)) \in \{5, 6\}$. Thus the relation "TC = Imm $-\,1$" holding for parallelizable real projective spaces no longer holds in the case of the other parallelizable flag manifolds $\mathbb{F}(1^k, 1)$. In general, flag manifolds (whether parallelizable of not) seem to have a larger TC than an Imm. For instance, the 7-dimensional flag manifold $\mathbb{F}(1, 1, 3)$ has $\mathrm{Imm}(\mathbb{F}(1, 1, 3)) = 10$ (see [27, 33]) whereas, according to Theorem 1.1 below, $\mathrm{TC}(\mathbb{F}(1, 1, 3)) \in \{13, 14\}$. Our results on these topics are presented in Chapter 3.

A second purpose of this thesis aims at exhibiting subtle but substantial differences between Farber's original concept and Rudyak's extended definition. The point is that, after an initial examination, Rudyak's higher TC could seem to be a close relative of Farber's TC. For instance, it has been shown that the families of spaces $X$ whose TC has been computed have an equally computable higher TC (cf. [2, 18, 20]). Likewise, some theoretical results for TC have reasonable (although sometimes more complicated to prove) higher TC generalizations, see for instance [5, 6, 21, 28]. However, other interesting theoretical properties known for TC do not have a known higher TC counterpart. For instance, it is known that the standard upper bound $2\dim(X)$ for $\mathrm{TC}(X)$ can be lowered by one unit whenever $\pi_1(X) = \mathbb{Z}_2$ ([7]). Before our studies, it was not even clear whether the proof of such a fact could be generalized to the higher TC. As a consequence of Theorem 1.2 below (with $k = 1$), we now know that such a potential $\mathrm{TC}_s$ generalization is doomed to fail for $s \geq 3$.

Closely related to the above fact is the phenomenon that there are infinite families of spaces for which the computation of their TC would require a non-elementary homotopy theoretic argument, but whose higher TC can be computed using purely cohomological methods. Indeed, as it is shown in this work (Theorem 1.2 and Proposition 3.12), flag manifolds $\mathbb{F}(1^k, 2^e - k + 1)$ with $k \leq 3$ have such a property[1]. Technically speaking, this phenomenon can be summarized by saying that, in many cases, *the* $\mathrm{TC}_2$*-obstruction described in [7, Theorem 7] vanishes without the vanishing of the analogous* $\mathrm{TC}_s$*-obstruction for* $s \geq 3$ (see the comments in Eq. 1.3, below). Results on this matter are enumerated in Chapter 3.

Another objective in this work, is the generalization of the relationship between TC and Imm holding for real projective spaces $\mathbb{F}(1, m)$. Namely, it is shown in [15] that, for the $m$-dimensional real projective space $\mathbb{R}\mathrm{P}^m$, $\mathrm{TC}_2(\mathbb{R}\mathrm{P}^m)$ agrees with $\mathrm{Imm}(\mathbb{R}\mathrm{P}^m)$, the Euclidean immersion dimension of $\mathbb{R}\mathrm{P}^m$, provided $m \neq 1, 3, 7$. Using the main result in [1], this means that, without restriction on $m$, $\mathrm{TC}_2(\mathbb{R}\mathrm{P}^m)$ can be described, in purely homotopic terms, as the minimal positive integer $a(m)$, also denoted by $\mathrm{axial}(\mathbb{R}\mathrm{P}^m)$, for which the restriction to $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ of the Hopf multiplication

$$\mu\colon \mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty \to \mathbb{R}\mathrm{P}^\infty$$

can be compressed to a map $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^{a(m)}$ —a so called (optimal) axial map. With this in mind, it is natural to ask for the (geometric and homotopic) properties of $\mathbb{R}\mathrm{P}^m$ encoded by the higher analogues $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)$. Such a task is addressed in this thesis and, in doing so, we are naturally lead to Davis' projective product space $\mathrm{P}_{\mathbf{m}_\mathrm{s}}$, introduced in [8], and defined as the orbit space of $(S^m)^{\times s}$ by the diagonal (antipodal) $\mathbb{Z}_2$-action; in Davis' notation, $\mathbf{m}_s$ stands for the $s$-tuple $(m, \ldots, m)$.

In slightly more detail, for $s \geq 2$, a natural generalization of the construction in [15, (4.2)] leads to

$$\mathrm{TC}_s(\mathbb{R}\mathrm{P}^m) \geq \mathrm{secat}(\pi_s), \tag{1.1}$$

where $\pi_s\colon \mathrm{P}_{\mathbf{m}_\mathrm{s}} \to (\mathbb{R}\mathrm{P}^m)^{\times s}$ is the "pivoted axial" $(\mathbb{Z}_2)^{\times(s-1)}$-principal bundle whose projection map is induced by the $s$-fold Cartesian power of the Hopf double cover $S^m \to \mathbb{R}\mathrm{P}^m$ (further details of this construction are reviewed in Chapter 4). The central result in [15] asserts that Eq. 1.1 is an equality for $s = 2$. The proof of such a fact is achieved by

(1) Connecting $\mathrm{secat}(\pi_2)$ to the existence of (optimal) axial maps

$$\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^{\mathrm{secat}(\pi_2)},$$

and then

[1] It might be the case that the restriction on $k$ can be removed—see the second half of Remark 3.8.

(2) Showing how (optimal) motion planners for $\mathbb{R}\mathrm{P}^m$ are encoded by such axial maps.

On the other hand, when $m$ is even, the validity of a $\mathrm{TC}_s$-generalization in Item (1) is hinted both by Proposition 4.4 below and by the cohomological calculations in Section 4.3. In particular, for $m$ even and $s$ large enough, we prove that equality holds in Eq. 1.1, and compute the resulting explicit value of $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)$, see Corollary 4.9 below. Those results are detailed in Chapter 4.

On the basis of our results, we conjecture that equality always holds in Eq. 1.1. This would yield a full generalization of Farber-Tabachnikov-Yuzvinsky's result to the higher TC. Proving equality in Eq. 1.1 seems to be inherently more complex when $s \geq 3$. See Remarks 4.3–4.5 for a discussion of why proving equality in Eq. 1.1 is elementary for $s = 2$, while the corresponding task for $s \geq 3$ becomes interestingly more intricate.

## 1.1 Statement of the Main Results

We next state our main results and explain how they fit within the introductory considerations above. Further comments will be given throughout the thesis.

**Theorem 1.1** (Corollary 3.10)**.** *Let $k$ and $m$ be positive integers, $\delta \in \{0, 1, \ldots, k-1\}$, and set $\epsilon = \min(\delta, 1)$ and $\alpha(r) = \max(0, r)$. If $e$ is a non-negative integer satisfying $2\delta \leq 2^e \leq m+\delta$, then*

$$(k - \delta + \epsilon)(2^{e+1} - 1) + \alpha((\delta - 1)(2^e - 1)) - \epsilon \leq \mathrm{TC}(\mathbb{F}(1^k, m)) \leq k(2m + k - 1). \tag{1.2}$$

Of course, the parameter $e$ should be taken as large as possible in order to get the full strength of Theorem 1.1. Two special cases (where the estimate in Eq. 1.2 has a gap of a unit) should be singled out from this result, namely:

- $\mathrm{TC}(\mathbb{F}(1, 2^e)) \in \{2^{e+1} - 1, 2^{e+1}\}$.
- $\mathrm{TC}(\mathbb{F}(1^2, 2^e - 1)) \in \{2^{e+2} - 3, 2^{e+2} - 2\}$.

Since $\mathbb{F}(1, 2^e)$ is the real projective space $\mathbb{R}\mathrm{P}^{2^e}$, the first situation is resolved by the well known equality

$$\mathrm{TC}(\mathbb{R}\mathrm{P}^{2^e}) = 2^{e+1} - 1 \tag{1.3}$$

(see [15]). It might seem reasonable to expect $\mathrm{TC}(\mathbb{F}(1^2, 2^e - 1)) = 2^{e+2} - 3$. In any case, proving (disproving) such an equality is equivalent to showing the triviality (non-triviality) of the homotopy obstruction described in [7, Theorem 7] for $X = \mathbb{F}(1^2, 2^e - 1)$. The relevance of such a task becomes apparent by noticing that, as a special case of Theorem 1.2 below, the $\mathrm{TC}_s$-analogue of the above homotopy obstruction does not vanish for these spaces when $s \geq 3$:

**Theorem 1.2** (Theorem 3.11 below)**.** *For positive integers $e$, $k$ and $s$ with $e \geq 1 + \lfloor \frac{k-1}{2} \rfloor$ and $k \leq 3 \leq s$, $\mathrm{TC}_s(\mathbb{F}(1^k, 2^e - k + 1)) = s \times \dim(\mathbb{F}(1^k, 2^e - k + 1))$.*

For instance, when $k = 1$, we get $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^{2^e}) = 2^e s$ if $s \geq 3$ and $e \geq 1$, which is certainly *not the case* for $s = 2$, as noted in Eq. 1.3. Restriction $e \geq 1$ is also needed as $\mathrm{TC}_s(S^1) = s - 1$ is well known ([31, Section 4]).

It should be noted that Theorem 1.2 can be thought of as a very distinguished manifestation of a more general phenomenon, namely: for fixed $k$ and $m$, the mod-2 cohomological estimates in this work for $\mathrm{TC}_s(\mathbb{F}(1^k, m))$ become sharper as $s$ increases. Such a point will be clarified and worked out in Section 3.2 of this thesis (Remark 3.7 and Corollary 3.15).

Other interesting (almost-sharp) estimates for the higher topological complexity of some semi-complete flag manifolds not considered in Theorem 1.2 are discussed in Section 3.2. All together, our results seem to point out to what could be the best estimate that purely cohomological methods can yield for the higher topological complexity of semi complete flag manifolds $\mathbb{F}(1^k, m)$ (see Remark 3.11).

Theorems 1.1 and 1.2 (and related results discussed in Section 3.2) are based on the identification of suitably long products of zero-divisors. The form of the required factors follows patterns that depend strongly on the value of $k$. The identification of such patterns is a major task in this work that has greatly benefited from the help of extensive computer calculations. On the other hand, the complexity of the calculations supporting Theorems 1.1 and 1.2 is in sharp contrast with the easy situation for the complex analogues $\mathbb{F}_{\mathbb{C}}(n_1, \ldots, n_\ell)$. The latter manifolds are 1-connected and symplectic (even Kähler), so their $s$-th topological complexity is well known (and easy to see) to agree with $s(\dim(\mathbb{F}_{\mathbb{C}}(n_1, \ldots, n_\ell)))/2$ (see [2, Corollary 3.15]).

Theorem 1.2, gives of course an infinite family of spaces for which $\mathrm{TC}_s$ accessibility contrasts with the hardness of the $\mathrm{TC}_2$ situation. It would be interesting to know if such a phenomenon holds for other families of spaces.

Section 2.5 addresses the problem of extending Farber-Tabachnikov-Yuzvinky's relationship between TC and $Imm$ to the $\mathrm{TC}_s$ realm. A key point is the identification of the map that classifies the covering projection $\pi_s$ in Eq. (1.1):

**Proposition 1.3** (Proposition 4.2). *For $1 \leq i \leq s$ let $p_i\colon (\mathbb{R}\mathrm{P}^m)^{\times s} \to \mathbb{R}\mathrm{P}^m$ be the i-th projection, $\xi_m \to \mathbb{R}\mathrm{P}^m$ be the Hopf bundle over $\mathbb{R}\mathrm{P}^m$, and $\mu_s\colon (\mathbb{R}\mathrm{P}^m)^{\times s} \to (\mathbb{R}\mathrm{P}^\infty)^{\times(s-1)}$ classify $\pi_s$. Then, for $1 \leq i \leq s-1$, the i-th component $\mu_{i,s}$ of $\mu_s$ classifies $p_i^*(\xi_m) \otimes p_s^*(\xi_m)$.*

The conclusion of Proposition 1.3 can of course be stated by saying that $\mu_{i,s}$ is homotopic to the composition of the projection $p_{i,s}\colon (\mathbb{R}\mathrm{P}^m)^{\times s} \to \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ onto the $(i,s)$ coordinates, the inclusion $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \hookrightarrow \mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty$, and the Hopf multiplication $\mu\colon \mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty \to \mathbb{R}\mathrm{P}^\infty$. The results in this thesis provide evidence toward the possibility that the covering projection $\pi_s$ plays a central role in the calculation of the higher topological complexity $\mathrm{TC}_s$ of $\mathbb{R}\mathrm{P}^n$. More explicitly, we have the next conjecture

**Conjecture 1.1** (Conjecture 4.1). *An s-motion planning algorithm for $\mathbb{R}\mathrm{P}^m$ having* $\mathrm{secat}(\pi_s)+1$ *s-local rules can be constructed out of a map $\phi_s$ satisfying the condition in Eq.* (4.6)*. Consequently* $\mathrm{secat}(\pi_s) \geq \mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)$*, and Eq. 1.1 becomes an equality for any $s \geq 2$.*

The conjecture is motivated by its case $s = 2$ which holds true —see Proposition 4.4 and Remark 4.3 below. Corollary 4.9 in Section 4.3 is meant to gather evidence for the plausibility of Conjecture 1.1. A few additional instances where Conjecture 1.1 holds true are included latter in this work.

The next result provides further evidence toward Conjecture 1.1. Here $G_s = \mathbb{Z}_2^{s-1}$, $J_k(G_s)$ stands for the $(k+1)$-iterated self join power of $G_s$, and $U_j \subset J_k(G_s)$ consists of the barycentric expressions $\sum_{\ell=0}^{k} t_\ell g_\ell$ with $t_j > 0$.

**Proposition 1.4** (Proposition 4.4). *Let $D_s = \{(x_1, \ldots, x_s) \in (S^m)^{\times s} : x_i = x_s$ for some $i \in \{1, \ldots, s-1\}\}$. The conclusions in Conjecture 4.1 hold true if one starts with a $G_s$-equivariant map $\phi_s\colon (S^m)^{\times s} \to J_{\mathrm{secat}(\pi_s)}(G_s)$ satisfying Eq. 4.4 together with one of the following conditions:*

1. *For* every $j \in \{0, 1, \ldots, \mathrm{secat}(\pi_s)\}$*, $\phi_s(D_s)$ intersects at most a single component of $U_j$.*
2. *For* some $j_0 \in \{0, 1, \ldots, \mathrm{secat}(\pi_s)\}$*, $\phi_s(D_s)$ is fully contained in some component of $U_{j_0}$.*

The study about $\mathrm{TC}_s(F(1^k, m))$ and the study about the extending the relationship between TC and $Imm$ to the $\mathrm{TC}_s$ give us ours principal results. Furthermore these antecedents give different lines of research in a future, for one hand find the value of $\mathrm{TC}_s$ from infinity families from another flag manifolds or more evidence (or the proof) about the conjecture 4.1.

## 1.2 Publications

All these results gave us the possibility to write two articles that have been published, namely:

- González,J., Gutiérrez, B., Gutiérrez, Darwin., and Lara, A. Motion Planing in Real Flag Manifolds. *Homotopy, Homology and Aplications*, Vol 18(2), 2016, pp 359-375.
- Cadavid-Aguilar, Natalia.,Gonález J., Gutiérrez, Darwin., Guzmán-Sáenz A., Lara, A. Sequential motion planing algorithms in projective spaces: An approach to their immersion dimension. *Forum Mathematicum.* Vol 30, 2017, pp 269-295.

These two journals are indexed in the Journal Citation Reports (JCR).

Chapter 2

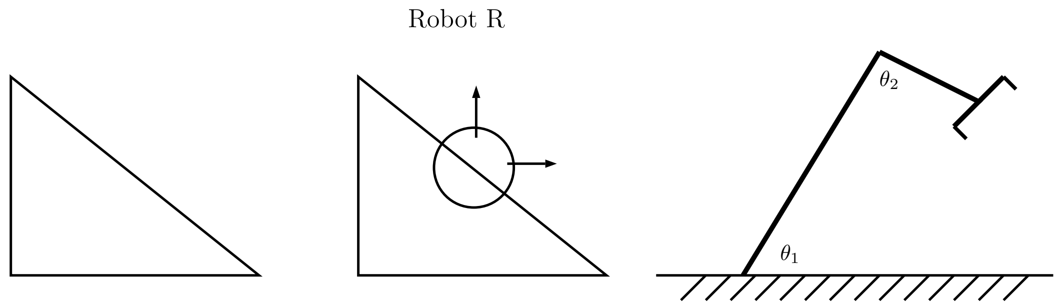
# State of the Art

In this chapter, the main topological concepts on the research of TC are summarized. This includes the properties and bounds on the TC parameter, results on the real flag manifolds and finally, the relation between TC and immersions.

## 2.1 Topological Complexity (TC)

This chapter starts with a concrete revision on the concepts behind the TC notion, and Farber's results obtained in the area of topological complexity. Some standard concepts and results from algebraic topological are stated without proof: they are our main tools for the work on this thesis. The main TC-results presented in this chapter are the following: first, an upper bound on the number $\mathrm{TC}(X)$ is given in terms of the *dimension* of the configuration space $X$ (Theorem 2.4), and on the other hand, a lower bound in terms of the structure of the cohomology ring of $X$ is presented (Theorem 2.5). The homotopy invariance of TC is also discussed, as well as the higher TC analogues of these properties.

Farber was the first researcher to work with a topological model for the motion planning problem in robotics. He introduced the concept of topological complexity $\mathrm{TC}(X)$ for a configuration space $X$, namely, $\mathrm{TC}(X)$ is a number which the end-points evaluation map measures the discontinuity of the process of motion planning in the space $X$. That is, $\mathrm{TC}(X)$ is defined as the minimal number $k$, such that $X \times X$ can be covered by $k + 1$ open subsets on each of which the end points evaluation map $P(X) \to X \times X$ admits a continuous section (also called a motion planning rule).




(a) Translate (b) Translate + rotate (c) A robotic arm R with 2 joints

Figure 2.1: Examples of Robots

The following discussion is meant to motivate the above Definition. Let $X$ be the space of all possible configurations of a mechanical system. A configuration can be informally understood as everything that is needed to describe where a robot is (assuming known kinematics, that is, its geometry). In most applications, the configuration space $X$ is also a topological space [12]. In 2.1, there are three examples of configurations, namely:

(a) A configuration for the robot in figure (2.1$a$) is determined by a pair $A = (x, y)$, i.e., the $x$ and $y$ coordinates are sufficient to describe the robot since it is restricted to translation motion. In this case $X = \mathbb{R}^2$.

(b) A configuration for the robot in figure (2.1$b$) is determined by three parameters $A = (x, y, \theta)$, i.e., in addition to the $(x, y)$ coordinates, we require an additional $\theta$ coordinate to specify the rotation. In this case $X = \mathbb{R}^2 \times S^1$ a cylinder.

(c) For the robotic arm $R$ in figure (2.1$c$), we require two parameters $\theta_1, \theta_2$ to completely specify the position of the arm in the world. Therefore, $A = (\theta_1, \theta_2)$ and in this case $X = S^1 \times S^1$ a torus.

Then, the **motion planning problem** consists in constructing a program or a devise, which takes pairs of configurations $(A, B) \in X \times X$ as an input and produces as an output, a continuous path in $X$, which starts at $A$ and ends at $B$. Here $A$ is the initial configuration, and and $B$ is the final (desired) configuration state of the system. As we see can in Figure 2.2 a change of states is equivalent to a path in the configuration space.

In the following, it is assumed that the configuration space $X$ is **path-connected**, which means that for any pair of points in $X$, there exists a continuous path in $X$ connecting them. In spite of this, in non-path-connected spaces only the path-connected components

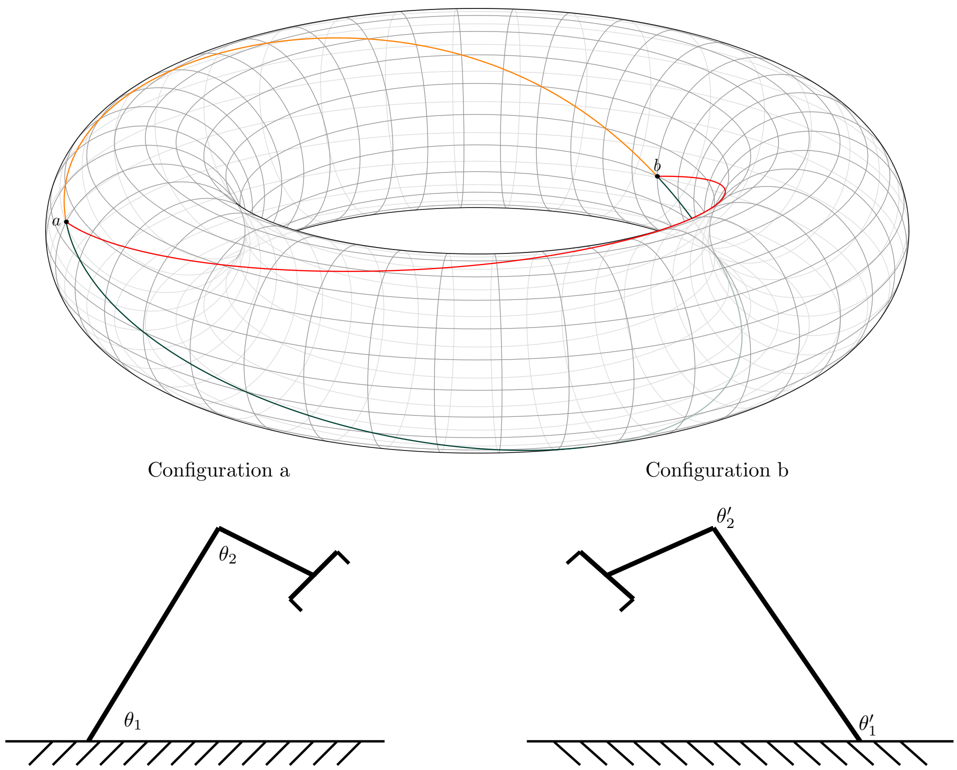



Figure 2.2: Motion planing on the torus

are considered; in a such case, the motion planner should decide first whether the given points $A$ and $B$ belong to the same path-connected component of $X$.

The motion planning problem is formalized as follows. Let $P(X)$ be the space of all continuous paths $\gamma : [0,1] \to X$, and $e : P(X) \to X \times X$ be the mapping which associates to any path $\gamma \in P(X)$ with the pair of initial and end points $e(\gamma) = (\gamma(0), \gamma(1))$. Additionally, the path space $P(X)$ is assumed to have the compact-open topology. Rephrasing the previous Definition, it is easy to see that the problem of motion planning in $X$ consists of finding a continuous function $s : X \times X \to P(X)$, such that the composition $e \circ s = id$ is the identity map. In other words, $s$ must be a **section** of $e$ i.e $e \circ s = Id_{X \times X}$.

At this point, a natural question arises: does there exist a continuous motion planning in $X$? Or equivalently, whether is it possible to construct a motion planning in the configuration space $X$, so that the continuous path $s(A, B)$ in $X$ (which describes the movement of the system from the initial configuration $A$ to final configuration $B$) depends continuously on the pair of points $(A, B)$?

Then, the aim in this theory, is to find a motion planning in $X$ such that the section $s : X \times X \to P(X)$ be continuous. Continuity in motion planning is an important natural requirement. Absence of continuity could result on instability of the desired behavior: in a such case, there will exist arbitrarily close pairs $(A, B)$ and $(A', B')$ of initial desired configurations such that the corresponding paths $s(A, B)$ and $s(A', B')$ are not close (in the topological sense). The existence of a global continuous section gives a strong restriction on the space $X$, as it is shown in the next result.

**Theorem 2.1.** *A continuous motion planning $s : X \times X \to P(X)$ exists if and only if the configuration space $X$ is contractible.*

*Proof.* Let us assume that there exist a continuous section, namely:

$$s : X \times X \to P(X).$$

Fix a point $A_0 \in X$ and consider the homotopy $h_t : X \to X$ given as $h_t(B) = s(A_0, B)(t)$, where $B \in X$, and $t \in [0, 1]$. So, $h_1(B) = B$ and $h_0(B) = A_0$. Therefore, $h_t$ gives a contraction of the space $X$ into the point $A_0 \in X$.

Now let us assume that there is a continuous homotopy $h_t : X \to X$ such that for any $A \in X$, $h_0(A) = A$ and $h_1(A) = A_0$. Then, for a pair $(A, B) \in X \times X$, the path $t \to h_t(A)$ can be concatenated with the reversal of the path $t \to h_t(B)$, to give a continuous motion planning in $X$. We obtain a motion planning in the contractible space $X$ by first moving $A$ into the base point $A_0$, along the contraction and then, following with the inverse of the path that brings $B$ to $A_0$. □

**Definition 2.1.** *Given a path-connected topological space $X$, the* **topological complexity** *of the motion planning in $X$ is defined as the minimal number $\mathrm{TC}(X) = k$, such that the product $X \times X$ can be covered by $k+1$ open subsets, that is, $X \times X = U_0 \bigcup U_1 \bigcup \cdots \bigcup U_k$, in such a way that for any index $i = 0, 1, 2, \cdots, k$ there exists a continuous motion planning $s_i : U_i \to P(X)$, with $\pi \circ s_i = id$ on $U_i$. Otherwise, the topological complexity is defined as $\mathrm{TC}(X) = \infty$.*

Note that the topological complexity $\mathrm{TC}(X)$ measures the discontinuity of any motion planner in $X$, moreover, it is and invariant up to homotopy (as it can be seen in the next section). Taken into account this information, next step is to define a motion planning algorithm.

**Definition 2.2.** *Let $\{U_i\}_{i=1,\ldots,k}$ be an open cover and $s_i$ be sections. The idea of a* **motion planning algorithm** *is organized as follows: given a pair of initial configurations $(A, B)$,* *we first find the subset $U_i$ with the smallest index $i$ such that $(A, B) \in U_i$; then consider the path $s_i(A, B)$ as an output and this a planning motion algorithm.*

As a consequence of Theorem 2.1, given a configuration space $X$, then $\mathrm{TC}(X) = 0$ if and only if $X$ is contractible.

Example **2.1.** *Suppose that $X$ is a convex subset of an Euclidean space $R^n$. Given a pair of initial-desired configurations $(A, B)$, we may move with constant velocity along the straight line segment connecting $A$ and $B$. This clearly produces a continuous algorithm for the motion planning problem in $X$. This is consistent because we have $\mathrm{TC}(X) = 0$ since $X$ is contractible.*

Example **2.2.** *Consider the case when $X = S^1$ namely, a circle. Since $S^1$ is not contractible, it is known that $\mathrm{TC}(S^1) \geq 1$. Let us show that $\mathrm{TC}(S^1) = 2$. Define $U_0 \subset S^1 \times S^1$ as $U_0 = \{(A, B) \mid A \neq -B\}$. A continuous motion planning over $U_0$ is given by the map $s_0 : U_0 \to P(S^1)$ which moves $A$ towards $B$ with constant velocity along the unique shortest arc-connecting $A$ to $B$. This map $s_0$ can not be extended to a continuous map on the pairs of antipodal points $A = -B$. Now define $U_1 = \{(A, B) \mid A \neq B\}$. Fix an orientation of the circle $S^1$. A continuous motion planning over $U_1$ is given by the map $s_1 : U_1 \to P(S^1)$ which moves $A$ towards $B$ with constant velocity in the positive direction along the circle. Again, $s_1$ cannot be extended to a continuous map on the whole $S^1 \times S^1$ and this construction is explicit.*

### 2.1.1 Homotopy Invariance of TC

Next property of homotopy invariance often allows a simplification on the configuration space $X$ without changing the topological complexity $\mathrm{TC}(X)$.

**Theorem 2.2.** $\mathrm{TC}(X)$ *depends only on the homotopy type of $X$.*

*Proof.* Suppose that $X$ dominates $Y$, i.e. there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$. Let us show that then $\mathrm{TC}(Y) \leq \mathrm{TC}(X)$. Assume that $U \subset X \times X$ is an open subset such that there exists a continuous motion planning $s : U \to P(X)$ over $U$. Define $V = (g \times g)^{-1}(U) \subset Y \times Y$. We will construct a continuous motion planning $\sigma : V \to P(Y)$ over $V$ explicitly. Fix a homotopy $h_t : Y \to Y$ with $h_0 = id_Y$ and $h_1 = f \circ g$; here $t \in [0, 1]$. For $(A, B) \in V$ and $\tau \in [0, 1]$ set

1. $\sigma(A, B)(\tau) = h_{3\tau}(A)$ if $\tau \in [0, \frac{1}{3}]$,

2. $\sigma(A,B)(\tau) = f(s(gA, gB)(3\tau - 1))$ if $\tau \in [\frac{1}{3}, \frac{2}{3}]$,

3. $\sigma(A,B)(\tau) = h_{3(1-\tau)}(B)$ if $\tau \in [\frac{2}{3}, 1]$.

Thus we obtain that for $k = \mathrm{TC}(X)$ any open cover $U_0 \cup \cdots \cup U_k = X \times X$ with a continuous motion planning over each $U_i$ defines an open cover $V_0 \cup \cdots \cup V_k = Y \times Y$ with the similar properties. This proves that $\mathrm{TC}(Y) \leq \mathrm{TC}(X)$, and obviously implies the statement of the Theorem. □

### 2.1.2 Bounds on $\mathrm{TC}(X)$

In previous section it was shown the problematic to give an explicit construction of continuous motion planning, and what is more, trying to prove that this construction be optimal. This is the reason why it is important to have tools to provide bounds on this invariant.

**Theorem 2.3.** *For any para-compact, path-connected space $X$,*

$$\mathrm{TC}(X) \leq 2dim(X).$$

*In particular, if $X$ is a connected polyhedral subset of $\mathbb{R}^n$ then the topological complexity* $\mathrm{TC}(X)$ *can be estimated from above as follows:*

$$\mathrm{TC}(X) \leq 2n - 2.$$

*Proof.* Details of this fact are in [12] Theorem 5. □

In order to provide examples of lower bounds, it is convenient to introduce the relationship between $\mathrm{TC}(X)$ and the Lusternik-Schnirelman category $\mathrm{cat}(X)$. Recall that $\mathrm{cat}(X)$ (in algebraic topology) is defined as the smallest integer $k$ such that $X$ may be covered by $k+1$ open subsets $V_0 \cup \cdots \cup V_k = X$, with each inclusion $V_i \to X$ being null-homotopic. Next result gives us lower bounds on the TC parameter.

**Theorem 2.4.** *If $X$ is path-connected and para-compact then*

$$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq 2\,\mathrm{cat}(X).$$

*Proof.* Let $U \subseteq X \times X$ be an open subset such that there exists a continuous motion planning $s : U \to P(X)$ over $U$. Let $A_0 \in X$ be a fixed point. Denote by $V \subseteq X$ the set of all points $B \in X$ such that $(A_0, B)$ belongs to $U$. Then clearly the set $V$ is open and it is contractible in $X$. This clearly leads to the first inequality in the conclusion of the Theorem. The second inequality is an immediate consequence of the subadditivity of $cat$ (i.e $cat(X \times Y) \leq cat(X) + cat(Y)$) and the obvious fact that the sectional category of a fibration $p : E \to B$ is bounded from above by the $LS$-category of the base space $B$. □

### Zero-divisors and TC

There is cohomological information that gives lower bounds for $\mathrm{TC}(X)$. Actually each generalized cohomology theory gives one such lower bound. In this section we show a relation between cohomology of $X$ an his $\mathrm{TC}(X)$. Let $K$ be a field. The cohomology $H^*(X; K)$ is a graded $K$-algebra with the multiplication

$$\cup : H^*(X; K) \otimes H^*(X; K) \to H^*(X; K),$$

given by the cup-product. The tensor product $H^*(X; k) \otimes H^*(X; k)$ is also a graded $K$-algebra with the multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = -1^{|v_1||u_2|} u_1 u_2 \otimes v_1 v_2.$$

Note that the cup-product is an algebra homomorphism.

**Definition 2.3.** *The kernel of the cup-product multiplication* $\cup$ *will be called the ideal of zero-divisors of* $H^*(X; K)$*. The zero-divisors-cup-length of* $H^*(X; K)$ *is the length of the longest non-trivial product in the ideal of zero-divisors of* $H^*(X; K)$

For instance, the zero-divisors cup-length of $H^*(S^n; \mathbb{Q})$ equals 1 for $n$ odd and 2 for $n$ even, the elements can be given explicitly.

**Theorem 2.5.** *The topological complexity of motion planning* $\mathrm{TC}(X)$ *is greater than or equal to the zero-divisors-cup-length of* $H^*(X; K)$

*Proof.* The topological complexity of motion planning $\mathrm{TC}(X)$ is the Schwartz genus of the fibration $e : P(X) \to X \times X$ and this map induces a homomorphism in cohomology:

$$e^* : H^*(X \times X; K) \to H^*(P(X); K),$$

Which, in terms of the Kunnet isomorphism $H^*(X \times X : K) \cong H^*(X : K) \otimes H^*(X : K)$ and the homotopy equivalence $P(X) \simeq X$, agrees with the cup product. Theorem follows from the cohomological lower bound for the Schwarz genus, see Theorem 4 of [12]. $\square$

Those bounds can be used to calculate TC for some well-known spaces, as it is shown in the following examples.

■ Example **2.3.** *The topological complexity of motion planning on the $n$-dimensional sphere $S^n$ is given by $\mathrm{TC}(S^n) = 1$ when $n$ is odd and $\mathrm{TC}(S^n) = 2$ when $n$ is even. This result holds since it is possible to give an explicit motion planning for the sphere having 2 and 3 open sets, respectively, and in view of the remark following Definition 2.3 above.*

■ Example **2.4.** *Let $X = \Sigma_g$ be a compact orientable two-dimensional surface of genus $g$. Then $\mathrm{TC}(\Sigma_g) = 2$ if $g \leq 1$, while $\mathrm{TC}(\Sigma_g) = 4$ if $g \geq 2$. These values follow from considerations with zero-cup-length in these spaces, and the same techniques will be used in the case of (semicomplete) flag manifolds, one of the central goals in this thesis,*

*Let us consider the case $g \geq 2$. Then, it is clear that there exist cohomology classes $u_1$, $v_1$, $u_2$, $v_2 \in H^1(\Sigma_g; \mathbb{Q})$ forming a symplectic system, i.e $u_i^2 = v_i^2 = 0$ and $u_1 v_1 = u_2 v_2 = A \neq 0$, and where $A \in H^2(\Sigma_g; \mathbb{Q})$ is the fundamental class, and besides $v_i u_j = v_i v_j = u_i u_j = 0$ for $i \neq j$. Then in the algebra $H^*(\Sigma_g; Q) \otimes H^*(\Sigma_g; Q)$ holds*

$$\prod_{i=1}^{2} (1 \otimes u_i - u_i \otimes 1)(1 \otimes v_i - v_i \otimes 1) = 2A \otimes A \neq 0.$$

*Thus, $\mathrm{TC}(\Sigma_g) \geq 4$ holds. The equality follows since $\mathrm{TC}(X) \leq 2dim(X)$.*

*In case $g = 0$, $\mathrm{TC}(\Sigma_g) = 2$. This is straightforward since $\Sigma_g = S^2$. Finally, for $g = 1$, it corresponds to the two-dimensional torus $T^2$, for which with cohomological arguments and the next result we have $\mathrm{TC}(T^2) = 2$.*

The number TC has properties that can be useful when we are working with more than one space at the same time.

**Theorem 2.6.** *For any path–connected metric spaces $X$ and $Y$,*

$$\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y).$$

*Proof.* Details of the poof are in [12]. $\square$

These notes and properties will be used in the next chapter in order to bound TC in flag manifolds, finding thus tight values for the number TC.

## 2.2 Higher Topological Complexity $\mathrm{TC}_s$

The idea of topological complexity was generalized by Yuli Rudyak in [31] in 2010. He defined the $s$-th topological complexity of $X$, $\mathrm{TC}_s(X)$, as the reduced Schwarz genus of the $s$-th fold evaluation map $e_s\colon P(X) \to X^s$ given by:

$$e_s(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{s-1}\right), \gamma\left(\frac{2}{s-1}\right), \dots, \gamma\left(\frac{s-2}{s-1}\right), \gamma(1)\right).$$

In particular, it holds $\mathrm{TC} = \mathrm{TC}_2$. Rudyak's *higher* topological complexity has been studied systematically in [2].

### 2.2.1 Properties of $\mathrm{TC}_s$

Many of the $\mathrm{TC}_s$ results generalize existing properties for Farber's TC (Section 2.1). For instance, the next result shows a close connection between higher topological complexity and the Lusternik-Schnirelmann category of Cartesian powers of spaces:

**Theorem 2.7.** *For a path-connected space $X$,* $\mathrm{cat}(X^{n-1}) \leq \mathrm{TC}_n(X) \leq \mathrm{cat}(X^n)$.

The proof of Theorem 2.7 is given later in this section. It is possible to prove that $\mathrm{TC}_n(G) = \mathrm{cat}(G^{n-1})$ for a path-connected topological group $G$, which extends the $n = 2$ property proved by Farber. Lupton and Scherer have recently proved that this property extends to not-necessarily associative Hopf spaces.

**Theorem 2.8.** *Let $f \times f_0 : X \times X_0 \to Y \times Y_0$ be the product of two maps $f : X \to Y$ and $f_0 : X_0 \to Y_0$. If $Y \times Y_0$ is normal, then $genus(f \times f_0) \leq genus(f) + genus(f_0)$.*

*Proof.* This result is proved in [2] □

It has been proved in [2] that $TC_n(X)$ can also be defined as follows:

**Definition 2.4.** *Let $X$ be a path-connected space. The $n$-**th topological complexity** of $X$, $\mathrm{TC}_n(X)$, is the Schwarz genus of the fibration*

$$e_n^X = e_n : X^{J_n} \to X^n, \quad e_n(\gamma) = (\gamma(1_1), \cdots, \gamma(1_n)), \tag{2.1}$$

*where $J_n$ is the wedge of $n$ closed intervals $[0,1]$ (each with $0 \in [0,1]$ as the base point), and $1_i$ stands for $1$ in the $i-th$ interval.*

Note that Eq. 2.1 is the standard fibrational substitute for the iterated diagonal map $d_n = d_n^X : X \to X^n$, so $\mathrm{TC}_n(X) = genus(d_n^X)$. More generally, for a contractible space $Y_n$ with $n$ distinct distinguished points $v_1, \cdots, v_n \in Y_n$, consider the evaluation map $e_{Y_n} : X^{Y_n} \to X^n, e_{Y_n}(f) = (f(y_1), \cdots, f(y_n))$. Because of the contractibility of $Y_n$, the genus of $e_{Y_n}$ is equal to $\mathrm{TC}_n(X)$, the proof is just as the one in [31]. In particular, we can take $Y_n$ to be a tree with $n - leaves$, or (as done at the beginning of this section) the unit interval $I_n$, say with distinguished points $v_i = (i-1)/(n-1), i = 1, \cdots, n$. In the later case we see that the $n-th$ higher topological complexity gives a topological measure of the complexity of the motion planning problem where the robot is required to visit $n$ ordered prescribed stages. For this reason, we also refer to $\mathrm{TC}_n$ as the $n-th$ sequential topological complexity. Farber's TC is $\mathrm{TC}_2$.

Other fibrations (which not necessarily give fibrational substitutes of the iterated diagonal) can be used to define $\mathrm{TC}_n$. Indeed, let $G_n$ be any connected graph where $n$ ordered distinct vertices $v_1, \cdots, v_n$ have been selected. We assert that the evaluation map $e_{G_n} : X^{G_n} \to X^n$ at the chosen vertices has genus $genus(e^{G_n}) = \mathrm{TC}_n(X)$. To see this, choose maps $I_n \to G_n \to J_n$ preserving the selected vertices. For instance, the latter map can be taken so to collapse most of $G_n$ to the base point in $J_n$, except that the first half of each directed edge $(v_i, v)$ in $G_n$ is mapped linearly onto the directed edge $(1_i, 0)$ in $J_n$ (in particular vertices $v_i$ are mapped to vertices $1_i$). Since the induced maps $X^{J_n} \to X^{G_n} \to X^{I_n}$ are compatible with the three evaluation maps, we get $genus(e_{I_n}) \leq genus(e_{G_n}) \leq genus(e_{J_n})$. But, as explained in the paragraph above, the extremes in the preceding chain of inequalities agree with $\mathrm{TC}_n(X)$.

The higher topological complexities of a space $X$ are closely related to the category of Cartesian powers of $X$. The first indication of such a property comes from the inequality $\mathrm{TC}_n(X) \leq \mathrm{cat}(X^n)$ which is an immediate consequence of the well known fact that the Schwarz genus of a fibration does not exceed the category of the base space. On the other hand, the inequality $\mathrm{cat}(X) \leq \mathrm{TC}_2(X)$ in Theorem 2.4, and can be generalized to:

**Theorem 2.9.** *For any path-connected space $X$, $\mathrm{cat}(X^{n-1}) \leq \mathrm{TC}_n(X)$.*

*Proof.* Let $\mathrm{TC}_n(X) = k$ and choose a covering $B_0, B_1, \cdots, B_k = X^n$ such that there is a continuous section $s_i$ for $e_n^X$ over $B_i$ for $i = 0, \cdots, k$. Let $p : X^n \to X$ be the projection onto the first factor, choose $x_1 \in X$, and put $A_i = p^{-1}(x_1) \cap B_i$. Note that $\{A_i\}_{i=0}^k$, is an open cover for $p^{-1}(x_1)$. Since $p^{-1}(x_1)$ is homeomorphic to $X^{n-1}$, it suffices to show that each $A_i$ is contractible within $p^{-1}(x_1)$.

For a point $(x_1, x_2, \cdots, x_n) \in A_i$ consider the $n$ paths $\gamma_1, \cdots, \gamma_n$ making up the multipath $s_i(x_1, x_2, \cdots, x_n) = \{\gamma_j\}_{j=1}^n$. Then $\gamma_j(1) = x_j$ and $\gamma_j(0) = x_0$ for some $x_0 \in X$ which is independent of $j \in \{1 \cdots n\}$. Then, the constant path $\delta_1$ at $x_1$, and the paths $\delta_j$ $(j = 2, \cdots, n)$—formed by using the time reversed path $\gamma_j^{-1}$ the first half of the time, and $\gamma_1$ the second half—are the components of a path $\delta = (\delta_1, \cdots, \delta_n)$ in $p^{-1}(x_1)$ from $\delta(0) = (x_1, x_2, \cdots, x_n)$ to $\delta(1) = (x_1, x_1, \cdots, x_1)$. The continuity of $s_i$ implies that $\delta$ depends continuously on $(x_1, x_2, \cdots, x_n)$, so we have constructed a contraction of $A_i$ to $(x_1, x_1, ..., x_1)$ in $p^{-1}(x_1)$. Thus, $\mathrm{cat}(X^{n-1}) \leq \mathrm{TC}_n(X)$. □

Using the fact that $\mathrm{cat}(X^n) \leq n$ if $X$ is not contractible, we see that Theorem 2.9 recovers [[31], Proposition 3.5]. Note in addition that Theorem 2.7 follows from the above discussion.

Now, we define a function $zcl$ as in the precedent section. Given a space $X$ and a positive integer $n$, $zcl(X, n)$ denotes the cup-length of elements in the kernel of the map induced in cohomology by $d_n^X$. Thus, $zcl(X, n)$ is the largest integer $m$ for which there exist cohomology classes $u_i \in H(X^n; A_i)$ such that $d_n^*(u_i) = 0$ for $i = 1, \cdots, m$ and $u_1 \cup \cdots \cup u_m \neq 0 \in H(X^n; A_1 \otimes \cdots \otimes A_m)$. In the following result, which is proved in [2], $\mathrm{TC}_n(X)$ is lower bounded by $zcl(X, n)$, and upper bounded by the ratio between the connectivity $conn(X)$ and the homotopy dimension $hdim(X)$ of $X$; the latter being the smallest dimension of $CW$-complexes having the homotopy type of $X$.

**Theorem 2.10.** *For any path-connected space $X$ we have the inequalities*

$$zcl(X, n) \leq \mathrm{TC}_n(X) \leq \frac{hdim(X) \cdot n}{conn(X) + 1}.$$

*Proof.* The details appear in [2]. □

**Theorem 2.11.** *Let $X$ be a $CW$ complex of finite type, and $R$ a principal ideal domain. Take $u \in H^d(X^n; R)$ with $d > 0$ even, and assume that the $n$-fold iterated self $R$-tensor product $u^m \otimes \cdots \otimes u^m \in H^{md}(X; R)^{\otimes n}$ is an element of infinite additive order. Then $\mathrm{TC}_n(X) \geq mn$.*

*Proof.* For $i = 1, \cdots, n$, let $p_i : X^n \to X$ be the projection onto the $i^{th}$ factor and put $u_i = p_i^*(u) \in H^d(X^n, R)$. In view of Theorem 2.10, the required inequality follows from

$$v := (u_2 - u_1)^{2m}(u_3 - u_1)^m \ldots (u_n - u_1)^m \neq 0.$$

In order to check this, note that $v$ comes from the tensor product, which injects into the cohomology of the Cartesian product by the Künneth Theorem (this is where the finiteness hypotheses are used). So, calculations can be performed in the former $R$-module. Now, assuming that $dim(X) \leq dm + 1$, we have

$$\begin{aligned} v &= (u_2 - u_1)^{2m}(u_3 - u_1)^m \ldots (u_n - u_1)^m \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m (u_3 - u_1)^m \ldots (u_n - u_1)^m \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m u_3^m (u_4 - u_1)^m \ldots (u_n - u_1)^m \\ &= \cdots \\ &= (-1)^m \binom{2m}{m} u_1^m u_2^m \cdots u_n^m \end{aligned}$$

which is non-zero by hypothesis. On the other hand, for $dim(X)$ arbitrary, consider the skeletal inclusion $j : X^{dm+1} \to X$ and note that $v \neq 0$ since $j^*(v) \neq 0$. This proves the Theorem. □

**Corollary 2.12.** *For every closed simply connected symplectic manifold $M^{2m}$, $\mathrm{TC}_n(M) = nm$.*

*Proof.* This follows from Theorem 2.11 (taking $u$ to be the cohomology class given by the symplectic 2-form on $M$, and noting that the hypothesis on $u^m \otimes \cdots \otimes u^m$ holds since the coefficients are taken over the reals), the product inequality for category, and the inequality $\mathrm{cat}(M^{2m}) \leq m$ which follows from $\mathrm{cat}(M^{2m}) = m$, a well known fact. □

Of course, Corollary 2.12 applies to complex projective spaces. In the quaternionic case essentially the same proof gives:

**Corollary 2.13.** *The quaternionic projective space of real dimension $4m$, $\mathbb{H}P^m$, has $\mathrm{TC}_n(\mathbb{H}P^m) = nm$.*

Note that the previous corollaries imply that the upper bound in Corollary 2.12 are optimal in general.

## 2.3 Real Flag Manifolds

A real flag manifold is the quotient space $G/P$ of a real, connected, semisimple Lie group $G$ and a subgroup $P$ of a special type which is called a parabolic subgroup. Classical examples are the projective space and the Grassmann manifold. Topological properties of (real) flag manifolds have been a research topic for a long time. In the 1930's Ehresmann [11] computed the fundamental and homology groups of some classical (i.e. $G = SL(n, \mathbb{R})$) flag manifolds and slightly more recently Kocherlakota [24] computed the homology groups of real flag manifolds in terms of the Dynkin diagram of $G$.

The basic reference for the results reviewed in this section is Borel's fundamental paper [3]. In this thesis we focus on the classical case of Real Flag Manifolds, which arise as follows. Let $n_1, \cdots, n_r$ be positive integers and let $n = n_1 + \cdots + n_r$. The flag manifold $F(n_1, \cdots, n_r)$ is the homogeneous space $O(n)/O(n_1) \times \cdots \times O(n_r)$. Note that the underlying set of $F(n_1, \cdots, n_r)$ consists of those tuples $(V_1, \cdots, V_r)$ of vector subspaces $V_i$ of $\mathbb{R}^n$ which are mutually orthogonal and have $dim(V_i) = n_i$. For $i = 1, \cdots, r$, there is a tautological vector bundle $\gamma_i$ over $F(n_1, \cdots, n_r)$ whose total space is the subspace of $F(n_1, \cdots, n_r) \times \mathbb{R}^n$ consisting of the tuples $(V_1, \cdots, V_r, x)$ with $x \in V_i$. Note that the Whitney sum $\bigoplus_{i=1}^r \gamma_i$ is naturally isomorphic to the $n$ dimensional trivial vector bundle. In particular, by taking total Stiefel-Whitney classes, we get the relation

$$W(\gamma_1)W(\gamma_2)\cdots W(\gamma_r) = 1, \tag{2.2}$$

in the mod-2 cohomology of $F(n_1, \cdots, n_r)$. Here $W(\gamma_i) = 1 + w_{i1} + \cdots + w_{ij}$ where $w_{ij}$ stands for the $j$-th Stiefel-Whitney class of $\gamma_i$.

## 2.4 Cohomology of $F(n_1, \cdots, n_r)$

Borel showed that the mod-2 cohomology of $F(n_1, \cdots, n_r)$ is generated by the classes $w_{ij}$ subject solely to the several homogeneous relations encoded in Eq. 2.2. Namely:

**Theorem 2.14** (Borel)**.** *The mod-2 cohomology ring of $F(n_1, \cdots, n_r)$ is the polynomial ring over $\mathbb{Z}_2$ generated by the Stiefel-Whitney classes $w_{ij} \in H^j(F(n_1, \cdots, n_r); \mathbb{Z}_2)$ divided out by the ideal generated by the positive dimensional homogeneous components of $\Pi_{i=1}^r W(\gamma_i)$*

■ Example **2.5.** *The 1-dimensional component of $W(\gamma_1)W(\gamma_2)\cdots W(\gamma_r) = (1 + w_{11} + w_{12} + \cdots)(1 + w_{21} + w_{22} + \cdots)\cdots(1 + w_{r1} + w_{r2} + \cdots)$ is given by $w_{11} + w_{21} + \cdots + w_{r1}$, which is trivial in $H^1(F(n_1, \cdots, n_r); \mathbb{Z}_2)$. This means that the polynomial generator $w_{r1}$ is really superfluous in the presentation of the Theorem above. This observation will be brought to its last consequences latter in this thesis in the case of semicomplete flag manifold $F(1, \cdots, 1, m)$, whose cohomology mod-2 is recorded next.*

Thus in the particular case,

**Theorem 2.15.** *For the special case* $\mathbb{F}(1^k, n) = \mathbb{F}(1, \cdots, 1, n)$ *with* $k$*-ones the cohomology of these spaces is given by Borel's (non-minimal) presentation of* $H^*(\mathbb{F}(1^k, n); \mathbb{Z}_2)$*, namely it has generators* $x_i$ *and* $w_j$ *with the single (non-homogeneous) relation*

$$\sum_{j\geq 0} w_j \prod_{i=1}^{k}(1 + x_i) = 1.$$

In this Theorem, $x_i$ stands for the first Stiefel-Whitney class of $\gamma_i$, while $1 + w_1 + w_2 + \cdots + w_n$ stands for the total Stiefel-Whitney class of $\gamma_{k+1}$.

This presentation does not provide efficient calculations, so in Chapter 3 we will give a better presentation than this, where calculations will be simpler.

## 2.5 Relation between $\mathrm{TC}(\mathbb{R}\mathrm{P}^n)$ and $\mathrm{Imm}(\mathbb{R}\mathrm{P}^n)$

A main goal in this thesis is to generalize a classic result from Farber, Yuzvinsky and Tabachnikov. In this section we recapitulate some important results concerning the $\mathrm{TC} - Imm$ relationship indicated in the introduction.

We will make a revision of principal results from [15] because in Chapter 4 we will establish a relationship between $\mathrm{TC}_s$ and a generalization of Immersion.

We study case $X = \mathbb{R}\mathrm{P}^n$, i.e. the problem of computing the topological complexity of the real projective space $\mathrm{TC}(\mathbb{R}\mathrm{P}^n)$. This problem is much harder than finding the topological complexity of the complex projective space. We review the work of Farber, Tabachnykov and Yuzvinsky [15], the problem of finding the number $\mathrm{TC}(\mathbb{R}\mathrm{P}^n)$ is equivalent to the problem of finding the smallest $k$ such that $\mathbb{R}\mathrm{P}^n$ can be immersed into the Euclidean space $\mathbb{R}^k$.

We begin this section by proving a general result relating the topological complexity of a topological space with the Schwarz genus of a covering. Let $X$ be a finite-connected polyhedron space with an action of a discrete group $G$.

**Theorem 2.16.** *Let* $X$ *be a finite-connected polyhedron and let* $p : \widetilde{X} \to X$ *be a regular covering map with the group of covering transformations* $G$*. Let* $\widetilde{X} \times_G \widetilde{X}$ *be obtained from the product* $\widetilde{X} \times \widetilde{X}$ *by factorizing with respect to the diagonal action of* $G$*. Then, the topological complexity* $\mathrm{TC}(X)$ *of the space* $X$ *is greater than or equal to the Schwarz genus of the covering*

$$q : \widetilde{X} \times_G \widetilde{X} \to X \times X.$$

*Proof.* Remember that $e : P(X) \to X \times X$ will denote the canonical fibration of the space of paths $e(\gamma) = (\gamma(0), \gamma(1))$, where $\gamma \in P(X)$, $\gamma : [0,1] \to X$. Consider the following commutative diagram:

$$\begin{tikzcd} P(X) \arrow[rr, "f"] \arrow[dr, "e"'] & & \widetilde{X} \times_G \widetilde{X} \arrow[dl, "q"] \\ & X \times X, & \end{tikzcd}$$

where the map $f : P(X) \to \widetilde{X} \times_G \widetilde{X}$ is defined as follows: given a continuous path $\gamma : [0,1] \to X$, let $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ be any lift of $\gamma$, and we set $f(\gamma) = [(\widetilde{\gamma}(0), \widetilde{\gamma}(1))] \in \widetilde{X} \times_G \widetilde{X}$. The lift $\widetilde{\gamma}$ of $\gamma$ depends on the choice of the initial point $\widetilde{\gamma}(0) \in \widetilde{X}$ but nevertheless the map $f$ is well defined and continuous.

If $U$ is an open subset of $X \times X$ and $s : U \to P(X)$ is a continuous section of the fibration $e$ over $U$, then $f \circ s$ is a continuous section of $q$ over $U$.

If there exits an open covering $U_0 \cup \cdots \cup U_k$ of $X \times X$ with a continuous section $s_i$ of $e$ over each open set $U_i$, then $f \circ s_i$ is a continuous section of $q$ over $U_i$ and we see that the Schwarz genus of the covering $q$ is at most $k$. $\square$

**Corollary 2.17.** *The number* $\mathrm{TC}(\mathbb{R}\mathrm{P}^n)$ *is greater than or equal to the Schwarz genus of the two–fold covering* $S^n \times_{\mathbb{Z}_2} S^n \to \mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$.

We present this result in a different form.

If $n$ is fixed, we always denote by $\xi$ the canonical real line bundle over $\mathbb{R}\mathrm{P}^n$. The exterior tensor product $\xi \otimes \xi$ is a real line bundle over $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$. Its first Stiefel-Whitney class is $w_1(\xi \otimes \xi) = \alpha \times 1 + 1 \times \alpha \in H^1(\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$, where $\alpha \in H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ is the generator. This last condition determines uniquely the bundle $\xi \otimes \xi$.

**Corollary 2.18.** *The topological complexity* $\mathrm{TC}(\mathbb{R}\mathrm{P}^n)$ *is not less than the minimal $k$ such that the Whitney sum $(k+1)(\xi \otimes \xi)$ of $k+1$ copies of $\xi \otimes \xi$ admits a nowhere vanishing section.*

*Proof.* By Corollary 2.17 $\mathrm{TC}(\mathbb{R}\mathrm{P}^n)$ is not less than the Schwarz genus of the unit sphere bundle $q$ of $\xi \otimes \xi$. By a Theorem of Schwarz [32], the latter coincides with the smallest $k$ such that the $(k+1)$-fold fiberwise join $q * q * \cdots * q$ admits a section. But, clearly, the $(k+1)$-fold fiberwise join $q * q * \cdots * q$ coincides with the unit sphere bundle of the Whitney sum $(k+1)(\xi \otimes \xi)$. This implies our statement. $\square$

We have seen also $\mathrm{cat}(X) \leq \mathrm{TC}(X)$ thus,

$$n \leq \mathrm{TC}(\mathbb{R}\mathrm{P}^n).$$

Indeed the equality holds only at cases 1, 3, 7.

**Theorem 2.19.** *If* $n \geq 2^{r-1}$, *then* $\mathrm{TC}(\mathbb{R}\mathrm{P}^n) \geq 2^r - 1$

*Proof.* Let $\alpha \in H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ be the generator. Then, $1 \otimes \alpha + \alpha \otimes 1$ is a zero-divisor, and we consider its power $(1 \otimes \alpha + \alpha \otimes 1)^{2^r-1}$. The binomial expansion of this class contains the term $\binom{2^r-1}{n}\alpha^k \otimes \alpha^n$ where $k = 2^r - 1 - n < n$. It is well known that the binomial coefficients $\binom{2^r-1}{i}$ are odd for all $i$. Hence $\binom{2^r-1}{n}\alpha^k \otimes \alpha^n$ is a non-zero term, and so $(1 \otimes \alpha + \alpha \otimes 1)^{2^r-1}$ does not vanish either. Applying the results about zcl, one finds that the topological complexity of $\mathbb{R}\mathrm{P}^n$ is not less than $2^r - 1$. $\square$

### 2.5.1 Nonsingular Maps and Axial Maps

In this section, we recall some notions and basic results concerning nonsingular maps $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ and axial maps $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$. These maps appear in the mathematical literature in relation to the immersion problem $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$. The presentation of this section is based on section 5 in [15].

**Definition 2.5.** *A continuous map*

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$$

*is called nonsingular if it has the following two properties:*

*(a)* $f(\lambda u, \mu v) = \lambda\mu f(u, v) \;\; \forall u, v \in \mathbb{R}^n$ *and* $\lambda, \mu \in \mathbb{R}$

*(b)* $f(u, v) = 0$ *implies* $u = 0$ *or* $v = 0$

*Bilinear nonsingular maps give immersions of projective spaces into Euclidean space [16].*

We will see that the nonsingular maps in the sense of Definition 2.5 provide a convenient tool for constructing explicit motion planning algorithms in projective spaces.

As an illustration, let us show that for any $n$ there exists a nonsingular map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n-1}$. One constructs it as follows. Fix a sequence $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1} : \mathbb{R}^n \to \mathbb{R}$ of linear functionals such that any $n$ of them are linearly independent. For $u, v \in \mathbb{R}^n$, the value $f(u, v) \in \mathbb{R}^{2n-1}$ is defined as the vector whose $j$-th coordinate equals the product $\alpha_j(u)\alpha_j(v)$, where $j = 1, 2, \cdots, 2n-1$. If $u \neq 0$, then at least $n$ among the numbers $\alpha_1(u), \cdots, \alpha_{2n-1}(u)$ are nonzero. Hence if $u \neq 0$ and $v \neq 0$, there exists $j$ such that $\alpha_j(u)\alpha_j(v) \neq 0$ and thus $f(u, v) \neq 0 \in \mathbb{R}^{2n-1}$.

**Lemma 2.20.** *There are no nonsingular maps $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ with $k < n$.*

*Proof.* We may apply the Borsuk-Ulam Theorem to the map $u \to f(u, v)$, where $v \neq 0$ is fixed and where $u$ varies on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. By the Borsuk-Ulam Theorem, $f(u, v) = f(-u, v)$ for some $u \in S^{n-1}$, but the latter also is $-f(u, v)$ and thus $f(u, v) = 0$. This gives a contradiction with the nonsingularity property. □

**Lemma 2.21.** *For $n$ equal to $1, 2, 4, 8$ there are nonsingular maps $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with the property that for any $u \in \mathbb{R}^n$ $u \neq 0$, the first coordinate of $f(u, u)$ is positive. And if $n$ is different from $1, 2, 4, 8$, there are no nonsingular maps $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.*

*Proof.* The complete proof is in [15]. □

Now we check the relationship between nonsingular maps and axial maps.

**Definition 2.6.** *Let $n$ and $k$ be two positive integers with $n < k$. A continuous map*

$$g : \mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$$

*is called axial of type $(n, k)$ if its restrictions to $* \times \mathbb{R}\mathrm{P}^n$ and $\mathbb{R}\mathrm{P}^n \times *$ are homotopic to the inclusion maps $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$.*

Here, $*$ denotes a base point of $\mathbb{R}\mathrm{P}^n$. Note that for $n < k$ any continuous map $h : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$ is either homotopically trivial or it is homotopic to the inclusion map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$. If $\alpha \in H^1(\mathbb{R}\mathrm{P}^k; \mathbb{Z}_2)$ denotes the generator, then $h^*(\alpha) \in H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ is either zero or equal to $\alpha$. The map $h$ is homotopically trivial if and only if $h^*(\alpha) = 0$. This shows that the property of the axial map $g$, can be equivalently stated by the formula $g^*(\alpha) = \alpha \times 1 + 1 \times \alpha$. This last condition fixes the homotopy type of a map $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty$and we are interested in finding the smallest $k$ such that this map can be factorized through the inclusion $\mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^\infty$. This discussion explains that there always exists an axial map $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^{2n}$. In fact, with some extra effort, one shows that there always exists an axial map $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^{2n-1}$.

**Lemma 2.22.** *Assume that $1 < n < k$. There exists a bijection between nonsingular maps $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ (viewed up to multiplication by a nonzero scalar) and axial maps*

$$\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k .$$

*Proof.* Given a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$, one defines $g : \mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$, where for $u, v \in S^n \subset \mathbb{R}^{n+1}$, the value $g(u, v)$ is the line through the origin containing the point $f(u, v) \in \mathbb{R}^{k+1}$. To show that $g$ is indeed axial, we fix $v \in S^n$ and vary only $u \in S^n$. We see that the obtained map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$ lifts to a map $S^n \to S^k$ given by $u \to f(u, v)$, and the relation $f(-u, v) = -f(u, v)$ implies that $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$ is not null-homotopic. Similarly, using $f(u, -v) = -f(u, v)$, we find that the restriction of $g$ onto $* \times \mathbb{R}\mathrm{P}^n$ is not null-homotopic.

Suppose now that we are given an axial map. Passing to the universal covers, we obtain a continuous map $\tilde{g} : S^n \times S^n \to S^k$ (defined up to a sign). As explained above, the axial property translates into $\tilde{g}(-u, v) = -\tilde{g}(u, v) = \tilde{g}(u, -v)$ for all $u, v \in S^n$. Now, we may define a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ by

$$f(u, v) = |u||v| \cdot \tilde{g}\left(\frac{u}{|u|}, \frac{v}{|v|}\right) \quad u, v \in \mathbb{R}^{k+1} - \{0\},$$

completing the proof. □

**Lemma 2.23.** *Suppose that for a pair of integers $1 < n < k$, there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$. Then, there exists a nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$, having the following additional property: for any non-zero $u \in \mathbb{R}^{n+1}$, the first coordinate of $f(u, u) \in \mathbb{R}^{k+1}$ is positive.*

*Proof.* Given a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$, consider the corresponding axial map $g : \mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$. The axial property implies that the restriction of $g$ onto the diagonal $\mathbb{R}\mathrm{P}^n \subset \mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$ is null-homotopic. Hence, we may find $g' \simeq g$ such that $g' : \mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^k$ is constant along the diagonal. Now, consider the nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ corresponding to $g'$. We see that for all $u \in \mathbb{R}^{k+1}$, the values $f(u, u) \in \mathbb{R}^{k+1}$ lie on a ray emanating from the origin. By performing an orthogonal rotation, we may assume that all nonzero vectors of this ray have positive first coordinates. This proves our claim. □

**Theorem 2.24.** *The number $\mathrm{TC}(\mathbb{R}\mathrm{P}^n)$ coincides with the smallest integer $k$ such that there exists a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$.*

*Proof.* Details are in [15]. □

The proof of this Theorem needs the relationship between the existence of a nowhere vanishing section for the vector bundle $(k+1)(\xi \otimes \xi)$ and the existence of a nonsingular map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$ ($k$ is the same), together with the fact that if there exists a nonsingular map $g : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$, where $n+1 < k$, then $\mathbb{R}\mathrm{P}^n$ admits a motion planner with $k$ local rules.

The result about the correlation between nonsingular maps $g : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$, and motion planners will be generalized in this thesis. In preparation for such a generalization we give full details of:

**Proposition 2.25.** *If there exists a nonsingular map $g : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$, where $n+1 < k$, then $\mathbb{R}\mathrm{P}^n$ admits a motion planner with $k$ local rules, that is,*

$$\mathrm{TC}(\mathbb{R}\mathrm{P}^n) \leq k - 1.$$

*Proof.* We start with the following observation. Let $\phi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a scalar continuous map such that $\phi(\lambda u, \mu v) = \lambda\mu\phi(u, v)$ for all $u, v \in \mathbb{R}^{n+1}$ and $\lambda, \mu \in \mathbb{R}$. Let $U_\phi \subset \mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$ denote the set of all pairs $(A, B)$ of lines in $\mathbb{R}^{n+1}$ such that $A \neq B$ and $\phi(u, v) \neq 0$ for some points $u \in A$ and $v \in B$. It is clear that $U_\phi$ is open. We claim that there exists a continuous motion planning strategy over $U_\phi$, that is, there is a continuous map $s$ defined on $U_\phi$ with values in the space of continuous paths in the projective space $\mathbb{R}\mathrm{P}^n$ such that, for any pair $(A, B) \in U_\phi$ the path $s(A, B)(t)$, $t \in [0, 1]$ starts at $A$ and ends at $B$. We may find unit vectors $u \in A$ and $v \in B$ such that $\phi(u, v) > 0$. Such pair $u, v$ is not unique; instead of $u, v$, we may take $-u, -v$. Note that both pairs $u, v$ and $-u, -v$ determine the same orientation of the plane spanned by $A, B$. The desired motion planning map $s$ consists in rotating $A$ toward $B$ in this plane, in the positive direction determined by the orientation. Assume now additionally that $\phi : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is positive in the following sense: for any $u \in \mathbb{R}^{n+1}, u \neq 0$, one has $\phi(u, u) > 0$. Then, instead of $U_\phi$, we may take a slightly larger set $U'_\phi \subset \mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$ which is defined as the set of all pairs of lines $(A, B)$ in $\mathbb{R}^{n+1}$ such that $\phi(u, v) \neq 0$ for some $u \in A$ and $v \in B$. Now, all pairs of lines of the form $(A, A)$ belong to $U'_\phi$. Then, for $A \neq B$, the path from $A$ to $B$ is defined as above (rotating $A$ toward $B$ in the plane, spanned by $A$ and $B$, in the positive direction determined by the orientation), and for $A = B$, we choose the constant path at $A$. Continuity is not violated. A vector-valued nonsingular map $f : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ determines $k$ scalar maps $\phi_1, \cdots, \phi_k : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ (the coordinates) and the above described neighborhoods $U_{\phi_i}$ cover the product $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$ minus the diagonal. Since $n+1<k$, we may use Lemma 2.21. Hence, we may replace the initial nonsingular map by such an $f$ that for any $u \in \mathbb{R}^{n+1}, u \neq 0$, the first coordinate $\phi_1(u,u)$ of $f(u,u)$ is positive. The open sets $U'_{\phi_1}, U_{\phi_2}, ..., U_{\phi_k}$ cover $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$. We have described explicit motion planning strategies over each of these sets. Therefore, $\mathrm{TC}(\mathbb{R}\mathrm{P}^n) \leq k-1$. $\square$

In Chapter 4 we will use these constructions in order to the generalization about the relationship between $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)$ and $axial(\mathrm{P}_{\mathbf{m}_s})$.

# Chapter 3

# Motion Planning in Real Flag Manifolds

As it has been said in Section 2.1, the value of TC by a space $X$ can be bounded for zcl and by using the dimension of the manifold. Those results will be used to give information on TC for some families of flag manifolds.

Unless otherwise is noted, all cohomology rings we deal with, have $\mathbb{F}_2$-coefficients by simplicity of the calculations. Remember that Theorem 2.15 gives a presentation for $H^*(\mathbb{F}(1^k, m))$, this presentation will be improved with the next result, it will be used to give explicit lower bounds of $\mathrm{TC}_2(\mathbb{F}(1^k, m))$ by using zcl.

**Proposition 3.1.** *Let $m \geq 1$. A minimal presentation for the ring $H^*(\mathbb{F}(1^k, m))$ is given by generators $x_i$, $1 \leq i \leq k$, all of dimension 1, subject to the relations*

$$h_{m+i}(x_1, \ldots, x_{k+1-i}) = 0, \quad 1 \leq i \leq k. \tag{3.1}$$

*A graded additive basis for $H^*(\mathbb{F}(1^k, m))$ is given by the monomials*

$$x(n_1, \ldots, n_k) := \prod_{i=1}^{k} x_i^{n_i} \tag{3.2}$$

*where $n_i \leq m + k - i$, for $i = 1, \ldots, k$.*

**Remark 3.1.** *The above presentation is a strong generalization of Theorem 2.15, the one given for complete flags—the latter one is not minimal. The direct proof below should be compared to [30], a paper devoted to the proof (using Gröbner bases) of Proposition 3.1.*

*Proof.* [Proof of Proposition 3.1] Let $e_i$ denote the $i$-th elementary symmetric polynomial, and $h_i$ denote the $i$-th complete symmetric polynomial for $i \geq 0$ (where $e_0 = h_0 = 1$). In both cases, the relevant variables will be explicitly indicated. For $i = 1, \ldots, k+1$, let $\gamma_i$ stand for the $i$-th tautological bundle on $\mathbb{F}(1^k, m)$, and set $x_i = w_1(\gamma_i)$ for $i \leq k$, and $w_j = w_j(\gamma_{k+1})$ for $j \geq 0$, the Stiefel-Whitney classes. Borel's (non-minimal) presentation of $H^*(\mathbb{F}(1^k, m))$ has generators $x_i$ and $w_j$ with the single (non-homogeneous) relation

$$\sum_{j \geq 0} w_j \prod_{i=1}^{k} (1 + x_i) = 1.$$

This expression's component in dimension $j > 0$ is

$$\sum_{0 \leq t \leq j} w_{j-t} e_t(x_1, \ldots, x_k) = 0. \tag{3.3}$$

In particular, for $j = 1$, we get $w_1 = e_1(x_1, \ldots, x_k) = h_1(x_1, \ldots, x_k)$. Assuming inductively that $w_\ell = h_\ell(x_1, \ldots, x_k)$ for $\ell < j$, Eq. 3.3 gives

$$w_j = \sum_{1 \leq t \leq j} h_{j-t}(x_1, \ldots, x_k) e_t(x_1, \ldots, x_k) = h_j(x_1, \ldots, x_k).$$

This uses the basic relation between elementary and complete symmetric polynomials

$$\sum_{t=0}^{j} (-1)^t e_t(x_1, \ldots, x_k) h_{j-t}(x_1, \ldots, x_k) = 0. \tag{3.4}$$

Therefore, the generators $w_j$ are superfluous and, since $w_j = 0$ for $j > m$, we get $h_{m+i}(x_1, \ldots, x_k) = 0$ for $i > 0$. This corresponds to Eq. 3.1 if $i = 1$, otherwise use

$$h_{m+i}(x_1, \ldots, x_k) = h_{m+i}(x_1, \ldots, x_{k-1}) + x_k h_{m+i-1}(x_1, \ldots, x_k) \tag{3.5}$$

to get $h_{m+i}(x_1, \ldots, x_{k-1}) = 0$ for $i > 1$. Iteration of this argument yields Eq. 3.1. Further, these equations can be used to write any power $x_i^\ell$ with $\ell > m + k - i$ in terms of powers $x_j^n$ with $j < i$ or $n < \ell$. This shows that the monomials in Eq. 3.2 are additive generators of $H^*(\mathbb{F}(1^k, m))$. On the other hand, the inclusion of the fiber in the total space of the fibration $\mathbb{F}(1^{k-1}, m) \to \mathbb{F}(1^k, m) \to \mathbb{R}\mathrm{P}^{m+k-1}$ is surjective in mod 2 cohomology. Therefore the corresponding $\mathbb{F}_2$-Serre spectral sequence has trivial coefficients and collapses from its second stage (cf. Theorem 4.4 in page 126 of [29, Part I]). An easy inductive argument [1] then

[1] Alternatively see [22, Corollary 9.5.15].

shows that the $\mathbb{F}_2$-Poincaré polynomial of $\mathbb{F}(1^k, m)$ is

$$P(x) = \prod_{i=1}^{k} \frac{1 - x^{i+m}}{1 - x}.$$

The proof is complete since $P(1) = \prod_{i=1}^{k}(m+i)$, which is the number of monomials in Eq. 3.2. $\square$

The relations in Eq. 3.1 are a distilled form of a more general (equivalent but non-minimal) set of relations: our proof gives in fact

$$h_{m+i}(x_1, \ldots, x_{k-j}) = 0 \quad \text{if} \quad i > j \geq 0. \tag{3.6}$$

(Alternatively, Eq. 3.6 is a consequence of Eq. 3.1 and the obvious inclusions $\mathbb{F}(1^k, m) \hookrightarrow \mathbb{F}(1^k, m+1) \hookrightarrow \mathbb{F}(1^k, m+2) \hookrightarrow \cdots$.) In addition, the obvious action of the symmetric group $\Sigma_k$ on (the cohomology of) $\mathbb{F}(1^k, m)$ implies that the relations in Eq. 3.6 extend to

$$h_{m+i}(x_{\ell_1}, \ldots, x_{\ell_{k-j}}) = 0$$

for any $1 \leq \ell_1 < \cdots < \ell_{k-j} \leq k$ with $0 \leq j < i \leq k$. For instance,

$$x_i^{m+k} = 0 \neq x_i^{m+k-1} \;\; \text{for any} \;\; i = 1, \ldots, k, \tag{3.7}$$

where the non-triviality of $x_i^{m+k-1}$ comes from Eq. 3.2. As noted in [30, Example 3.1], this recovers the calculation in [26] of the heights of the generators $x_i$'s. Proposition 3.1 also allows us to recover the calculation of $\text{cat}(\mathbb{F}(1^k, m))$ in [26] (we use the normalized version of the Lusternik-Schnirelmann category, so that a contractible space $X$ has $\text{cat}(X) = 0$):

**Corollary 3.2.** $\text{cat}(\mathbb{F}(1^k, m)) = \dim(\mathbb{F}(1^k, m)) = km + k(k-1)/2.$

*Proof.* It is well known that $km + k(k-1)/2 = \dim(\mathbb{F}(1^k, m)) \geq \text{cat}(\mathbb{F}(1^k, m))$. The latter term is bounded from below by the $\mathbb{F}_2$-cup-length of $\mathbb{F}(1^k, m)$ which, in view of Proposition 3.1, is no less than $km + k(k-1)/2$ since $x_1^{m+k-1} x_2^{m+k-2} \cdots x_k^m \neq 0$. $\square$

**Corollary 3.3.** *The annihilator of the (non-trivial) class*

$$x_1^{m+k-1} x_2^{m+k-2} \cdots x_k^m \in H^*(\mathbb{F}(1^k, m))$$

*is the maximal ideal* $H^{>0}(\mathbb{F}(1^k, m))$ *of positive-degree elements. More precisely,*

$$x_1^{m+k-1} x_2^{m+k-2} \cdots x_j^{m+k-j} x_j = 0$$

*for* $1 \leq j \leq k$.

*Proof.* Apply, inductively on $j$, the relation in Eq. 3.1 with $i = k - j + 1$. □

More important for our later purposes is the fact that, as indicated in the proof of Proposition 3.1, the extended relations in Eq. 3.6 can be used in an inductive way to write any polynomial in the $x_i$'s in terms of the basis in Eq. 3.2. We next show that the resulting process can be written down with a nice closed formula if certain basis elements are to be neglected.

Here is an explicit example of calculations. Consider $H^*(\mathbb{F}(1^3, 2)) \approx \mathbb{Z}_2[x_1, x_2, x_3]/\sim$, where in this case $\sim$ is given by the ideal generated by the relations $h_3(x_1, x_2, x_3) = 0$, $h_4(x_1, x_2) = 0$, $h_5(x_1) = x_1^5 = 0$. If we want make arithmetic in this space, we must use those relations. For instance, for the element $x_1^4 x_2^3 x_3^3 \in H^*(\mathbb{F}(1^3, 2))$ :

$$\begin{aligned}
x_1^4 x_2^3 x_3^3 &= x_1^4 x_2^3 [h_3(x_1, x_2, x_3) - x_3^3] \\
&= x_1^4 x_2^3 (x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_1 x_2 x_3 + x_2^2 x_3 + x_2 x_3^2 + x_2^3) \\
&= x_1^4 x_2^3 (x_2^2 x_3 + x_2 x_3^2 + x_2^3), \text{ since } x_1^5 = 0 \\
&= x_1^4 x_2^4 x_3^2, \text{ in view of } Eq.\ 3.8 \\
&= x_1^4 [h_4(x_1, x_2) - x_2^4] x_3^2 \\
&= x_1^4 [x_1 A] x_3^2 = x_1^5 A x_3^2 = 0, \text{ where} A = A(x_1, x_2).
\end{aligned}$$

In this example we see how the procedure is for the reduction of a monomial and this calculation can be programmed.

In this algorithm $p = x_1^4 x_2^3 x_3^3 \in H^*(\mathbb{F}(1^3, 2))$ returns $p' = 0$.

**Proposition 3.4.** *Let* $0 \leq j \leq i \leq k$ *with* $i \geq 1$. *In terms of the basis in Eq. 3.2, all basis elements* $x(n_1, \ldots, n_k)$ *appearing in the expression of*

$$x_i^{m+k-j} + x_i^{m+k-i} e_{i-j}(x_1, \ldots x_{i-1}) \in H^*(\mathbb{F}(1^k, m))$$

*have* $n_i < m + k - i$ *and* $n_\ell = 0$ *for* $\ell > i$.

**Algorithm 1:** Reduction

**Data:** $p = p(x_1, \cdots, x_k) \in H^*(\mathbb{F}(1^k, m))$
**Result:** $p' \sim p$ basic
$p' \leftarrow p$;
**while** $p'$ *is not basic* **do**
  Replace each power $x_i^{m-i+k+1}$ in $p'$ by $h_{m-i+k+1}(x_1, \ldots, x_i) - x_i^{m-i+k+1}$;
  $p' \leftarrow \text{ReducePoly}(p')$
**end**
**return** $p'$

*Proof.* The cases $j = 0$ and $j = i$ hold vacuously true in view of Eq. 3.7. The case $j = i - 1$ follows by observing that a repeated use of Eq. 3.5 allow us to write the relations in Eq. 3.1 as

$$x_i^{m+k-i+1} = x_i^{m+k-i} h_1 + x_i^{m+k-i-1} h_2 + \cdots + h_{m+k-i+1} \tag{3.8}$$

where the complete symmetric polynomials are evaluated at the variables $x_1, \ldots, x_{i-1}$. All other cases ($0 < j < i - 1$) follow from an obvious (decreasing) inductive calculation using Eq. 3.4 and the corresponding analogue of Eq. 3.8. □

## 3.1 $\mathbb{F}_2 - \mathrm{zcl}$ Bounds for $\mathrm{TC}(\mathbb{F}(1^k, m))$

Most of the existing methods to estimate the topological complexity of a given space are cohomological in nature and are based on some form of obstruction theory. For our purposes, the most successful method to estimate Farber's topological complexity are in Subsection 2.1.2, where we saw that $\mathrm{TC}_2$ is in between zcl and $2dim$.

We use the notation $\lambda_i$ (resp. $\rho_i$) for the generators $x_i$ on the left (resp. right) tensor factor of $H^*(\mathbb{F}(1^k, m) \times \mathbb{F}(1^k, m)) = H^*(\mathbb{F}(1^k, m)) \otimes H^*(\mathbb{F}(1^k, m))$. The sum $\lambda_i + \rho_i$, which is a zero-divisor, will be denoted by $z_i$.

Thus the cohomology ring $H^*(\mathbb{F}(1^k, m)) \otimes H^*(\mathbb{F}(1^k, m))$ its the polynomial ring

$$\mathbb{Z}_2[\lambda_1, \cdots, \lambda_k, \rho_1, \cdots, \rho_k]$$

with the relations $h_{m+i}(\lambda_1, \ldots, \lambda_{k+1-i}) = 0$, $h_{m+i}(\rho_1, \ldots, \rho_{k+1-i}) = 0$ with $1 \leq i \leq k$. And we look for $max\{\sum_{j=1}^{k} e_j\}$ such that $z_1^{e_1} \cdots z_k^{e_k} \in H^*(\mathbb{F}(1^k, m)) \otimes H^*(\mathbb{F}(1^k, m))$ is non-zero, where $z_i = \lambda_i + \rho_i$ . This work can be programmed and it gives indication of a general pattern. The algorithm shown in the table "Algorithm 2:ZCL" is similar to the reduction algorithm because we have the same relations, and we know $z_i^{m-1} \neq 0$.

**Algorithm 2:** ZCL

**Data:** $k, m$
**Result:** $(e_1, \ldots, e_k)$ s.t. maximizes $\{\sum_{j=1}^{k} e_j\}$
$e_j \leftarrow m-1$, $j = 1, \ldots, k$;
**while** $p = z_1^{e_1} \cdots z_k^{e_k} \neq 0$ $(z_j = \lambda_j + \rho_j)$ **do**
  Replace powers of $\lambda$ and $\rho$ in $p$ taken from relations
    $h_{m+i}(\lambda_1, \ldots, \lambda_{k+1-i}) = 0$, and
    $h_{m+i}(\rho_1, \ldots, \rho_{k+1-i}) = 0$;
  ReducePoly($z_1^{e_1} \cdots z_k^{e_k}$) ;
  PermuteValues($(e_1, \ldots, e_k)$) or IncreaseOne($\{\sum_{j=1}^{k} e_j\}$);
**end**
**return** $(e_1, \ldots, e_k)$

Algorithm 2 was used to make extensive calculations in certain important cases. That exercise gave us clues that point to the form of the solution for general case. Theory confirms evidence as we will show in next results. However, in practice, the computational technique has some problems due to the computational power needed in the polynomial arithmetic: it needs long space in memory to store information (in spite that it can use dynamic memory, and that the reductions are performed in every stage). Thus we need find patterns in "small" convenient cases.

**Lemma 3.5.** *In the ring $H^*(\mathbb{F}(1^k, 2^e))^{\otimes 2}$ we have*

$$(z_1 \cdots z_k)^{2^{e+1}-1} \neq 0. \tag{3.9}$$

**Remark 3.2.** *When $k \leq 2^e$, Eq. 3.9 is sharp in the sense that $z_j^{2^{e+1}} = \lambda_j^{2^{e+1}} + \rho_j^{2^{e+1}} = 0$ for $1 \leq j \leq k$, in view of Eq. 3.7. However, such an optimality in Eq. 3.9 is far from holding when $k > 2^e$. For instance, Eq. 3.9 asserts that $z_1 z_2 z_3 \neq 0$ in $H^*(\mathbb{F}(1^4))^{\otimes 2}$, but we will show in fact (Proposition 3.7 below) that $z_1^3 z_2^3 z_3^2 \neq 0$ in $H^*(\mathbb{F}(1^4))^{\otimes 2}$. A similar phenomenon holds for $\mathbb{F}(1^3, 2)$—replacing the use of Proposition 3.7 by Theorem 3.9 below (with $k = 3$).*

*Proof.* [Proof of Lemma 3.5] We proceed by induction on $k$. The case for $k = 1$ is elementary and well known—note that $\mathbb{F}(1, 2^e)$ is the real projective space $\mathbb{R}\mathrm{P}^{2^e}$. Assume the result is valid for $k$ and consider the fibration

$$\mathbb{F}(1^k, 2^e) \xrightarrow{\iota} \mathbb{F}(1^{k+1}, 2^e) \xrightarrow{\pi} \mathbb{F}(1, 2^e + k) = \mathbb{R}\mathrm{P}^{2^e+k} \tag{3.10}$$

where $\pi(L_1, \ldots, L_{k+1}, V) = (L_1, V \oplus \bigoplus_{2 \leq i \leq k+1} L_i)$. Since $\iota$ is surjective in cohomology, [29, Theorem 4.4] shows that the Serre spectral sequence for the term-wise cartesian square of Eq. 3.10 has a trivial system of coefficients, and collapses from its second term. The result follows since, by the inductive hypothesis, the left-hand side term in Eq. 3.9 is non-zero in the second stage of the spectral sequence. □

Theorem 2.10, Corollary 3.2, and Lemma 3.5 yield the estimate in Corollary 3.6 below for the topological complexity of manifolds $\mathbb{F}(1^k, m)$ admitting an equatorial inclusion $\mathbb{F}(1^k, 2^e) \hookrightarrow \mathbb{F}(1^k, m)$ with $2^e \leq m$.

**Corollary 3.6.** *Let $e$ denote the integral part of $\log_2(m)$. Then*

$$k(2^{e+1} - 1) \leq \mathrm{TC}(\mathbb{F}(1^k, m)) \leq k(2m + k - 1). \tag{3.11}$$

Note that the smallest gap in Eq. 3.11 is of $k^2$ units (for $m = 2^e$). On the other hand, the lower bound in Eq. 3.11 is optimal in general. For instance, the gap of a unit for $\mathrm{TC}(\mathbb{F}(1, 2^e))$ coming from Corollary 3.6 (with $k = 1$) is solved by Eq. 1.3. This is of course compatible with the first assertion in Remark 3.2. But for $k > 1$ there is room for improvements of the lower bound in Eq. 3.11 by a zero-divisors cup-length analysis of an intermediate space $\mathbb{F}(1^k, m')$ with $2^e < m' \leq m$ and $m'$ not a power of 2. For instance, since Eq. 3.7 yields $z_j^{2^{e+1}} = 0$ in $H^*(\mathbb{F}(1^2, 2^{e+1} - 2))^{\otimes 2}$, the only possibility to improve the lower bound in Eq. 3.11 for $k = 2$ via zero-divisors cup-length considerations can come only through the analysis of the case for $\mathbb{F}(1^2, 2^{e+1} - 1)$. In fact, we next give zero-divisors cup-length bounds for $\mathrm{TC}(\mathbb{F}(1^k, m))$ inherent to the case $m = 2^e - 1$. As observed at the end of Remark 3.3, our argument will apply only for $k \geq 2$ —after all, the case of $\mathbb{F}(1, 2^e - 1)$, the real projective space of dimension $2^e - 1$, has been one of the most difficult situations studied over the years (see for instance [9, 23]).

**Proposition 3.7.** *Assume $k \geq 2$ and $e \geq 1$. In $H^*(\mathbb{F}(1^k, 2^e - 1))^{\otimes 2}$ we have*

$$(z_1 \cdots z_{k-1})^{2^{e+1}-1} z_k^{2^{e+1}-2} \neq 0. \tag{3.12}$$

**Remark 3.3.** *When $k \leq 2^e + 1$, Eq. 3.12 is almost sharp in the sense that $z_j^{2^{e+1}} = 0$ for $1 \leq j \leq k$, in view of Eq. 3.7. Computer calculations show that such an optimality feature fails in general if $k > 2^e + 1$, for instance with complete Flag manifolds $\mathbb{F}(1^k, 1)$. Also worth* *noticing is that, in Remark 3.4 below, we give evidence suggesting*

$$(z_1 \cdots z_k)^{2^{e+1}-1} = 0$$

*in Proposition 3.7. On the other hand, note that Eq. 3.12 certainly fails for $k = 1$.*

*Proof.* [Proof of Proposition 3.7] The inductive argument in the proof of Lemma 3.5, now replacing Eq. 3.10 by the fibrations

$$\mathbb{F}(1^k, 2^e - 1) \to \mathbb{F}(1^{k+1}, 2^e - 1) \to \mathbb{F}(1, 2^e + k - 1) = \mathbb{R}\mathrm{P}^{2^e+k-1},$$

shows that the general case in Proposition 3.7 follows inductively from the case $k = 2$. On the other hand, for the latter case, Eq. 3.7 gives in $H^*(\mathbb{F}(1, 1, 2^e - 1))^{\otimes 2}$

$$\begin{aligned}
z_1^{2^{e+1}-1} z_2^{2^{e+1}-2} &= (\lambda_1 + \rho_1)^{2^{e+1}-1} (\lambda_2 + \rho_2)^{2^{e+1}-2} \\
&= \left(\lambda_1^{2^e} \rho_1^{2^e-1} + \lambda_1^{2^e-1} \rho_1^{2^e}\right) \left(\lambda_2^2 + \rho_2^2\right)^{2^e-1} \\
&= \left(\lambda_1^{2^e} \rho_1^{2^e-1} + \lambda_1^{2^e-1} \rho_1^{2^e}\right) \left((\lambda_2^2)^{2^{e-1}} (\rho_2^2)^{2^{e-1}-1} + (\lambda_2^2)^{2^{e-1}-1} (\rho_2^2)^{2^{e-1}}\right) \\
&= \left(\lambda_1^{2^e} \rho_1^{2^e-1} + \lambda_1^{2^e-1} \rho_1^{2^e}\right) \left(\lambda_2^{2^e} \rho_2^{2^e-2} + \lambda_2^{2^e-2} \rho_2^{2^e}\right).
\end{aligned}$$

Further, if $\mu$ stands for either $\lambda$ or $\rho$, the relations in Eq. 3.1 give $\mu_2^{2^e} = \mu_1^2 A_\mu + \mu_1 \mu_2^{2^e-1}$. Thus

$$\begin{aligned}
& z_1^{2^{e+1}-1} z_2^{2^{e+1}-2} \\
&= \left(\lambda_1^{2^e} \rho_1^{2^e-1} + \lambda_1^{2^e-1} \rho_1^{2^e}\right) \left(\lambda_2^{2^e} \rho_2^{2^e-2} + \lambda_2^{2^e-2} \rho_2^{2^e}\right) \\
&= \left(\lambda_1^{2^e} \rho_1^{2^e-1} + \lambda_1^{2^e-1} \rho_1^{2^e}\right) \left(\left(\lambda_1^2 A_\lambda + \lambda_1 \lambda_2^{2^e-1}\right) \rho_2^{2^e-2} + \lambda_2^{2^e-2} \left(\rho_1^2 A_\rho + \rho_1 \rho_2^{2^e-1}\right)\right) \\
&= \lambda_1^{2^e} \rho_1^{2^e-1} \cdot \lambda_2^{2^e-2} \rho_1 \rho_2^{2^e-1} + \lambda_1^{2^e-1} \rho_1^{2^e} \cdot \lambda_1 \lambda_2^{2^e-1} \rho_2^{2^e-2} \\
&= \lambda_1^{2^e} \lambda_2^{2^e-2} \rho_1^{2^e} \rho_2^{2^e-1} + \lambda_1^{2^e} \lambda_2^{2^e-1} \rho_1^{2^e} \rho_2^{2^e-2}.
\end{aligned}$$

The result follows as the two monomials in the last expression are basis elements. □

**Remark 3.4.** *Before discussing the implications of Proposition 3.7 to the topological complexity of flag manifolds, we make a brief pause to say a few words about the sharpness of Proposition 3.7 when $2 \leq k \leq 2^e + 1$ —hypothesis that will be in force in this paragraph. Since $0 = z_i^{2^{e+1}} \in H^*(\mathbb{F}(1^k, 2^e - 1))^{\otimes 2}$ for all $i$, the triviality of any product $z_{i_1} \cdots z_{i_t}$ with*

*$t \geq (2^{e+1}-1)k$ is equivalent to*

$$(z_1 \cdots z_k)^{2^{e+1}-1} = 0. \tag{3.13}$$

*Proving Eq. 3.13 presents a major challenge (not addressed in this work). Checking the validity of Eq. 3.13 for $k=2$ is easy in view of the last expression for $z_1^{2^{e+1}-1} z_2^{2^{e+1}-2}$ at the end of the proof of Proposition 3.7. We have checked the validity of Eq. 3.13 for $k \in \{3,4\}$ with the help of a computer, but the task quickly becomes computationally prohibitive for larger values of $k$ as the number of basis elements in the expression on the left-hand term of Eq. 3.12 increases extremely fast as $k$ grows: 16 basis elements are need for $k=3$, while the number of required basis elements increases to 1128 for $k=4$—the sum of which would have to vanish after multiplying by $z_k$, should Eq. 3.13 be true.*

**Corollary 3.8.** *Let $e$ denote the integral part of $\log_2(m+1)$. If $e \geq 1$ and $k \geq 2$, then*

$$k(2^{e+1}-1)-1 \leq \mathrm{TC}(\mathbb{F}(1^k, m)) \leq k(2m+k-1). \tag{3.14}$$

**Remark 3.5.** *The smallest gap in Eq. 3.14 is of $k(k-2)+1$ units (for $m = 2^e-1$). The case of $\mathbb{F}(1^2, 2^e-1)$ (e.g. the closed parallelizable 3-manifold $\mathbb{F}(1^3)$) is particularly appealing, as the corresponding gap is only of one unit.*

Example **3.1.** *It is elementary to see that, when $k=1$, Corollary 3.6 captures all the* TC*-information cohomologically available from Lemma 3.5. Indeed, the relations $z_1^{2^{e+1}-1} \neq 0 = z_1^{2^{e+1}}$, holding in $H^*(\mathbb{F}(1,2^e))^{\otimes 2}$, clearly hold in any $H^*(\mathbb{F}(1,m))^{\otimes 2}$ with $2^e \leq m < 2^{e+1}$. Similarly, Corollaries 3.6 and 3.8 capture all the* TC*-information available from Lemma 3.5 and Proposition 3.7 if $k=2$. For $z_1^{2^{e+1}-1} z_2^{2^{e+1}-2} \neq 0$ holds sharply in $H^*(\mathbb{F}(1,1,2^e-1))^{\otimes 2}$ (c.f. Remark 3.4), and $z_1^{2^{e+1}-1} z_2^{2^{e+1}-1} \neq 0$ holds sharply in $H^*(\mathbb{F}(1,1,m))^{\otimes 2}$ for $2^e \leq m \leq 2^{e+1}-2$ (c.f. Remark 3.2).*

Example 3.1 below (and extensive computer calculations) seem to suggest that all the zcl-information for $\mathbb{F}(1^k, m)$ is contained in the cases $m = 2^e - \delta$ with $0 \leq \delta < k$. The analysis of the corresponding zcl properties is the subject of the remainder of this section (see Theorem 3.9).

Example **3.2.** *Apply the inductive argument in the proofs of Lemma 3.5 and Proposition 3.7 to the fibration $\mathbb{F}(1, 2^e-2) \to \mathbb{F}(1^k, 2^e-2) \to \mathbb{F}(1^{k-1}, 2^e-1)$. For $k \geq 3$ and $e \geq 1$, Proposition 3.7 gives $(z_1 \cdots z_{k-2})^{2^{e+1}-1} z_{k-1}^{2^{e+1}-2} \neq 0$ in $H^*(\mathbb{F}(1^{k-1}, 2^e-1))^{\otimes 2}$. Since*

$0 \neq z_1^{2^e-1} \in H^*(\mathbb{F}(1,2^e-2))^{\otimes 2}$ *is a standard calculation for* $e \geq 2$, *we see*

$$(z_1 \cdots z_{k-2})^{2^{e+1}-1}\, z_{k-1}^{2^{e+1}-2} z_k^{2^e-1} \neq 0 \quad \textit{in } H^*(\mathbb{F}(1^k, 2^e-2))^{\otimes 2} \textit{ if } k \geq 3 \textit{ and } e \geq 2. \tag{3.15}$$

*What is remarkable in Eq. 3.15 is that, although this argument is really measuring the zero-divisors cup-length of a graded object associated to* $H^*(\mathbb{F}(1^k, 2^e-2))^{\otimes 2}$, *extensive computer calculations suggest that no cohomological information has been missed.*

**Remark 3.6.** *At a first glance, Eq. 1.3 and Remark 3.5 might suggest that the methods in this work could lead to estimate the* TC *of* $\mathbb{F}(1^3, 2^e-2)$ *with an error of at most a unit. The error, however, increases exponentially with e (see Example 3.3). Yet, as shown in Section 3.2 below, the corresponding* $\mathrm{TC}_s$ *estimates for* $s \geq 3$ *will in fact be sharp.*

The argument leading to Eq. 3.15 can be iterated with the fibrations

$$\mathbb{F}(1, 2^e-\delta) \to \mathbb{F}(1^k, 2^e-\delta) \to \mathbb{F}(1^{k-1}, 2^e-\delta+1)$$

for $\delta \geq 2$ (but note that the case $\delta = 1$ fails to recover Proposition 3.7) to get the following generalizations of Lemma 3.5 and Proposition 3.7, and of Corollaries 3.6 and 3.8:

**Theorem 3.9.** *The following assertions hold in* $H^*(\mathbb{F}(1^k, m))^{\otimes 2}$:

*(a) For* $m + k \leq 2^{e+1}$ *and* $1 \leq i \leq k$, $z_i^{2^{e+1}} = 0$.

*(b) For* $m = 2^e - \delta$ *with* $k > \delta \geq 0$ *and* $2^{e-1} \geq \delta$,

$$(z_1 \cdots z_{k-\delta})^{2^{e+1}-1}\, z_{k-\delta+1}^{2^{e+1}-2}\, (z_{k-\delta+2} \cdots z_k)^{2^e-1} \neq 0. \tag{3.16}$$

**Corollary 3.10.** *Let k and m be positive integers,* $\delta \in \{0, 1, \ldots, k-1\}$, *and set* $\epsilon = \min(\delta, 1)$ *and* $\alpha(r) = \max(0, r)$. *If a nonnegative integer e satisfies* $2\delta \leq 2^e \leq m + \delta$, *then*

$$(k-\delta+\epsilon)(2^{e+1}-1) + \alpha((\delta-1)(2^e-1)) - \epsilon \leq \mathrm{TC}(\mathbb{F}(1^k, m)) \leq k(2m+k-1). \tag{3.17}$$

Due to the form of the exponents of the factors on the left-hand side of Eq. 3.16, the gap in Eq. 3.17 becomes in general larger as the parameter $\delta$ increases. Still, as shown in the following examples, there are concrete situations where Corollary 3.10 yields better lower bounds for larger values of $\delta$.

■ Example **3.3.** *Obtaining the sharpest information from Theorem 3.9 and Corollary 3.10 for a fixed flag manifold* $\mathbb{F}(1^k, m_0)$ *usually requires choosing a suitable combination of parameters* $(e, \delta)$ *with* $2^e - \delta \leq m_0$ *(so that the non-triviality of a cohomology class in* $\mathbb{F}(1^k, m_0)$ *can be obtained, via Theorem 3.9, from the non-triviality of its restriction to* $\mathbb{F}(1^k, 2^e - \delta)$*). Take for instance the case of* $\mathbb{F}(1^3, 2)$ *where the conclusion of item (b) in Theorem 3.9 with* $\delta = 0$ *is* $(z_1 z_2 z_3)^3 \neq 0$*, but the conclusion with* $\delta = 2$ *is in fact* $z_1^7 z_2^6 z_3^3 \neq 0$*. Alternatively, the case* $\delta = 0$ *in Corollary 3.10 implies* $\mathrm{TC}(\mathbb{F}(1^3, 6)) \geq 21$*, while the case* $\delta = 2$ *yields the stronger estimate* $\mathrm{TC}(\mathbb{F}(1^3, 6)) \geq 36$*. In particular, the smallest gap in Eq. 3.17 for* $\mathbb{F}(1^3, 6)$ *is of 6 units and corresponds to* $\delta = 2$*. More generally, for* $e \geq 2$*, the smallest gap in Eq. 3.17 for* $\mathbb{F}(1^3, 2^e - 2)$ *is of* $2^e - 2$ *units. In particular* $\mathrm{TC}(\mathbb{F}(1^3, 2)) \in \{16, 17, 18\}$*—a gap of only two units.*

As indicated in Remarks 3.2 and 3.3, the lower bound in Eq. 3.17 tends to get weaker as $k$ is larger than $2^e - \delta$. Extreme cases hold (with $(e, \delta) \in \{(0, 0), (1, 1)\}$) for complete flag manifolds $\mathbb{F}(1^k, 1)$.

## 3.2 Higher Topological Complexity

This thesis has shown methods to give almost-sharp estimates for the topological complexity of flag manifolds $\mathbb{F}(1, 2^e)$ and $\mathbb{F}(1^2, 2^e - 1)$. This section's goal is to show that, in the realm of higher topological complexity, the cohomological estimates become sharp and, above all, valid for other flag manifolds of the form $\mathbb{F}(1^k, 2^e - k + 1)$. In general, our results show that, as $s$ increases, the cohomological method becomes better suited to estimate $\mathrm{TC}_s(\mathbb{F}(1^k, m))$. This point will be made precise in Remark 3.7 and Corollary 3.15 below.

As it has seen in Subsection 2.2.1 (Theorem 2.10), value of $\mathrm{TC}_s(X)$ is bounded as follows

$$zcl(X, n) \leq \mathrm{TC}_n(X) \leq \frac{hdim(X) \cdot n}{conn(X) + 1}. \tag{3.18}$$

In what follows we use the notation $zcl_n(X)$ as an alternative for $zcl(X, n)$

**Remark 3.7.** *Let* $G(k, m, s)$ *denote the gap in Eq. 3.21 and* $X = \mathbb{F}(1^k, m)$*, i.e.* $G(k, m, s) = s \cdot d_{k,m} - \mathrm{zcl}_s(\mathbb{F}(1^k, m))$ *where* $d_{k,m} = km + k(k-1)/2$ *(c.f. Corollary 3.2). Corollary 3.15 below indicates that, for k and m fixed, the sequence of non-negative integers* $\{G(k, m, s)\}_{s \geq 2}$ *is monotonically decreasing and, therefore, eventually constant. In fact, in the main result of this section (Theorem 3.11 below), the monotonic phenomenon holds with a zero limiting* *value,* $\lim_{s\mapsto\infty} G(k,m,s) = 0$*, thus getting sharp results in this particular case.*

**Theorem 3.11.** *For positive integers $e$, $k$ and $s$ with $e \geq 1 + \lfloor \frac{k-1}{2} \rfloor$ and $k \leq 3 \leq s$,* $\mathrm{zcl}_s(\mathbb{F}(1^k, 2^e - k + 1)) = \mathrm{TC}_s(\mathbb{F}(1^k, 2^e - k + 1)) = s \dim(\mathbb{F}(1^k, 2^e - k + 1))$.

Theorem 3.11 is an immediate consequence of Eq. 3.18 and the inequality $s\dim(\mathbb{F}(1^k, 2^e - k + 1)) \leq \mathrm{zcl}_s(\mathbb{F}(1^k, 2^e - k + 1))$, which will be established in Propositions 3.12 and 3.13 for $k \leq 3 \leq s$ by identifying non-trivial products with suitably many $s$-th zero-divisors as factors.

For $1 \leq i \leq s$ and $1 \leq j \leq k$, let $x_{i,j}$ be the pullback class $\pi_i^*(x_j) \in H^*(\mathbb{F}(1^k, m))^{\otimes s}$ where $\pi_i \colon \mathbb{F}(1^k, m)^s \to \mathbb{F}(1^k, m)$ is the $i$-th projection ($1 \leq i \leq s$ and $1 \leq j \leq k$), and let $z_{i,j}$ stand for the $s$-th zero-divisor $x_{1,j} + x_{i,j}$. We will deal with the basis of Eq. 3.2 and its tensor product basis

$$\prod_{i=1}^{s} x_i(n_{i,1}, \ldots, n_{i,k}), \quad 0 \leq n_{i,j} \leq m + k - j, \tag{3.19}$$

where $x_i(n_{i,1}, \ldots, n_{i,k}) = \pi_i^*(x(n_{i,1}, \ldots, n_{i,k}))$.

**Proposition 3.12.** *For $s \geq 3$ and $e \geq 1$,*

1. $z_{2,1}^{2^{e+1}-1} z_{3,1}^{2^e+1} z_{4,1}^{2^e} \cdots z_{s,1}^{2^e}$ *is non-trivial in the cohomology ring* $H^*(\mathbb{F}(1, 2^e))^{\otimes s}$.
2. $(z_{2,1}^{2^{e+1}-1} z_{2,2}^{2^{e+1}-2}) \cdot (z_{3,1}^{2^e-1} z_{3,2}^{2^e+1}) \cdot (z_{4,1}^{2^e} z_{4,2}^{2^e-1}) \cdots (z_{s,1}^{2^e} z_{s,2}^{2^e-1})$ *is non-trivial in the cohomology ring* $H^*(\mathbb{F}(1^2, 2^e - 1))^{\otimes s}$.

**Proposition 3.13.** *For $s \geq 3$ and $e \geq 2$,*

$$0 \neq (z_{2,1}^{2^{e+1}-1} z_{2,2}^{2^{e+1}-2} z_{2,3}^{2^e-1}) \cdot (z_{3,1}^{2^e-1} z_{3,2}^{2^e-1} z_{3,3}^{2^{e+1}-3}) \cdot \prod_{i=4}^{s} (z_{i,1}^{2^e} z_{i,2}^{2^e-1} z_{i,3}^{2^e-2})$$

*in the cohomology ring* $H^*(\mathbb{F}(1^3, 2^e - 2))^{\otimes s}$.

**Remark 3.8.** *Note that the powers of the factors $z_{2,j}$ in the three products above coincide with the relevant power(s) of the products in Eq. 3.9 with $k = 1$, Eq. 3.12 with $k = 2$, and Eq. 3.15 for $k = 3$. In the present case ($s \geq 3$), the form of the powers of the factors $z_{3,j}$ is what allows us to get sharp results. Also worth mentioning is the possibility that Theorem 3.11 could hold true by relaxing the restriction "$k \leq 3 \leq s$" to "$k \leq s$" (see Remark 3.11 for a more general possibility). The proof of such an assertion seems to require computational* *input (suggesting suitable generalizations of Propositions 3.12 and 3.13) that does not seem to be currently available with today's computer capabilities.*

Lemma 3.14 below implies that it suffices to prove Propositions 3.12 and 3.13 for $s = 3$.

**Lemma 3.14.** *For $2 \leq i \leq s$, the expression in $H^*(\mathbb{F}(1^k, m))^{\otimes s}$ of*

$$z_{i,1}^{m+k-1} z_{i,2}^{m+k-2} \cdots z_{i,k-1}^{m+1} z_{i,k}^{m} + x_i(m+k-1, m+k-2, \cdots, m+1, m)$$

*in terms of the basis in Eq. 3.19 involves only basis elements of the form*

$$x_1(r_1, \ldots, r_k) \cdot x_i(t_1, \ldots, t_k)$$

*with $r_j > 0$ for some $j \in \{1, \ldots, k\}$ (so $t_{j'} < m + k - j'$ for some $j' \in \{1, \ldots, k\}$).*

*Proof.* Expand out $z_{i,1}^{m+k-1} z_{i,2}^{m+k-2} \cdots z_{i,k-1}^{m+1} z_{i,k}^{m}$ and notice that all the resulting monomials are basis elements. □

**Corollary 3.15.** *For $k, m \geq 1$, $G(k, m, 2) \geq G(k, m, 3) \geq G(k, m, 4) \geq \cdots \geq 0$.*

*Proof.* If $z \in H^*(\mathbb{F}(1^k, m))^{\otimes s}$ is some non-trivial product of $s$-th zero-divisors, then Lemma 3.14 implies that $z \cdot z_{s+1,1}^{m+k-1} z_{s+1,2}^{m+k-2} \cdots z_{s+1,k-1}^{m+1} z_{s+1,k}^{m} \in H^*(\mathbb{F}(1^k, m))^{\otimes(s+1)}$ is non-trivial too. The result then follows from the bare definition of the function $G(k, m, s)$. □

Propositions 3.12 and 3.13 are proved by direct computation of the given products. In all cases, advantage is taken of the fact that the products lie in the top dimension $s(km + \binom{k}{2})$ of the relevant ring $H^*(\mathbb{F}(1^k, m))^{\otimes s}$, where the additive basis in Eq. 3.19 reduces to the single element

$$\prod_{i=1}^{s} x_i(m+k-1, m+k-2, \ldots, m+1, m). \tag{3.20}$$

Explicitly, we use the inductive process indicated in the proof of Proposition 3.1, except that, since Eq. 3.20 is the only basis element we care about, the extended relations in Eq. 3.6 can be replaced by the relations

$$x_i^{m+k-j} = x_i^{m+k-i} e_{i-j}(x_1, \ldots x_{i-1}), \text{ for } 0 \leq j \leq i \leq k \text{ and } i \geq 1, \tag{3.21}$$

coming from Proposition 3.4. Proof details for Proposition 3.12 are similar (and easier) than those for Proposition 3.13, so we only focus on the latter case.

*Proof.* [Proof of Proposition 3.13] By Lemma 3.14 (see also the proof of Corollary 3.15), we only need to consider the case $s = 3$. We will show that for $e \geq 2$,

$$(z_{2,1}^{2^{e+1}-1} z_{2,2}^{2^{e+1}-2} z_{2,3}^{2^e-1}) \cdot (z_{3,1}^{2^e-1} z_{3,2}^{2^e-1} z_{3,3}^{2^{e+1}-3}) = \prod_{i=1}^{3} x_{i,1}^{2^e} x_{i,2}^{2^e-1} x_{i,3}^{2^e-2}, \tag{3.22}$$

the top basis element in $H^*(\mathbb{F}(1^3, 2^e-2))^{\otimes 3}$.

The mod-2 arithmetic of binomial coefficients, and the fact that $x_{i,j}^{2^e+1} = 0$ give

$$z_{3,3}^{2^{e+1}-3} = (x_{1,3} + x_{3,3})^{2^{e+1}-3} = x_{1,3}^{2^e} x_{3,3}^{2^e-3} + x_{1,3}^{2^e-3} x_{3,3}^{2^e} \tag{3.23}$$

(of course, this uses the hypothesis $e \geq 2$). Due to the form of the relations in Eq. 3.21 —or Eq.3.6 for that matter— and since $x_{3,3}^{2^e-3}$ is a basis element, the term $x_{1,3}^{2^e} x_{3,3}^{2^e-3}$ above cannot contribute to the top basis element. In other words, the considerations around Eq. 3.20 imply that the product of the term $x_{1,3}^{2^e} x_{3,3}^{2^e-3}$ with the first five powers on the left of Eq. 3.22 vanishes. Such an argument will be used repeatedly in what follows, and will simply be referred to by using a "$\equiv$" symbol. In these terms, the relations in Eq. 3.21 allow us to extend Eq. 3.23 to $z_{3,3}^{2^{e+1}-3} = x_{1,3}^{2^e} x_{3,3}^{2^e-3} + x_{1,3}^{2^e-3} x_{3,3}^{2^e} \equiv x_{1,3}^{2^e-3} x_{3,3}^{2^e} \equiv x_{1,3}^{2^e-3} x_{3,1} x_{3,2} \cdot x_{3,3}^{2^e-2}$,

$$\begin{aligned}
z_{3,2}^{2^e-1} z_{3,3}^{2^{e+1}-3} &\equiv z_{3,2}^{2^e-1} x_{1,3}^{2^e-3} x_{3,1} x_{3,2} \cdot x_{3,3}^{2^e-2} \\
&= (x_{1,2} + x_{3,2})^{2^e-1} x_{1,3}^{2^e-3} x_{3,1} x_{3,2} \cdot x_{3,3}^{2^e-2} \\
&\equiv (x_{1,2} x_{3,2}^{2^e-2} + x_{3,2}^{2^e-1}) x_{1,3}^{2^e-3} x_{3,1} x_{3,2} \cdot x_{3,3}^{2^e-2} \\
&= x_{1,3}^{2^e-3} x_{3,1} (x_{1,2} + x_{3,1}) \cdot x_{3,2}^{2^e-1} x_{3,3}^{2^e-2},
\end{aligned}$$

and

$$\begin{aligned}
z_{3,1}^{2^e-1} z_{3,2}^{2^e-1} z_{3,3}^{2^{e+1}-3} &\equiv (x_{1,1} + x_{3,1})^{2^e-1} x_{1,3}^{2^e-3} x_{3,1} (x_{1,2} + x_{3,1}) \cdot x_{3,2}^{2^e-1} x_{3,3}^{2^e-2} \\
&\equiv x_{1,3}^{2^e-3} (x_{1,1} x_{3,1}^{2^e-2} + x_{3,1}^{2^e-1}) x_{3,1} (x_{1,2} + x_{3,1}) \cdot x_{3,2}^{2^e-1} x_{3,3}^{2^e-2} \\
&= x_{1,3}^{2^e-3} (x_{1,1} + x_{1,2}) \cdot x_{3,1}^{2^e} x_{3,2}^{2^e-1} x_{3,3}^{2^e-2}.
\end{aligned}$$

An entirely similar (and straightforward) calculation gives

$$z_{2,1}^{2^{e+1}-1} z_{2,2}^{2^{e+1}-2} z_{2,3}^{2^e-1} \equiv (x_{1,1}^{2^e} x_{1,2}^{2^e-2} x_{1,3} + x_{1,1}^{2^e-1} x_{1,2}^{2^e}) \cdot x_{2,1}^{2^e} x_{2,2}^{2^e-1} x_{2,3}^{2^e-2},$$

and the result then follows since an additional (and much simpler) such computation gives

$x_{1,3}^{2^e-3}(x_{1,1}+x_{1,2})\cdot(x_{1,1}^{2^e}x_{1,2}^{2^e-2}x_{1,3}+x_{1,1}^{2^e-1}x_{1,2}^{2^e}) \equiv x_{1,1}^{2^e}x_{1,2}^{2^e-1}x_{1,3}^{2^e-2}.$ □

The Serre spectral sequence method used in Section 3.1 could now be coupled with Propositions 3.12 and 3.13 to get an extension of Theorem 3.11 on the lines of Corollary 3.10. However such a task would need to be done in a carefully selective way as, in some cases, the direct computations in the previous proof give better results. In fact, as the following example suggests (see also the proof of Proposition 3.16), best results can be obtained by a suitable combination of both techniques.

Example **3.4.** *Proposition 3.12(1) and the Serre spectral sequence applied to the fibration* $\mathbb{F}(1,2^e)\to\mathbb{F}(1,1,2^e)\to\mathbb{F}(1,2^e+1)$ *(with* $e\geq 1$*) yield the non-triviality of*

$$(z_{2,1}^{2^{e+1}-1}z_{2,2}^{2^{e+1}-1})\cdot(z_{3,1}^{2^e+1}z_{3,2}^{2^e+1})\cdot(z_{4,1}^{2^e}z_{4,2}^{2^e})\cdots(z_{s,1}^{2^e}z_{s,2}^{2^e})\in H^*(\mathbb{F}(1,1,2^e))^{\otimes s}$$

*for* $s\geq 3$*. But one can do better. For instance, a direct argument (spelled out in Proposition 3.16 below) gives in fact the non-triviality of*

$$\left(z_{2,1}^3z_{2,2}^3\right)\cdot\left(z_{3,1}^3z_{3,2}^3\right)\cdot\left(z_{4,1}^3z_{4,2}^2\right)\cdots\left(z_{s,1}^3z_{s,2}^2\right)\in H^*(\mathbb{F}(1,1,2))^{\otimes s} \tag{3.24}$$

*for* $s\geq 3$*, so that* $5s-3\leq \mathrm{TC}_s(\mathbb{F}(1,1,2))\leq 5s$*. As a result we have that* $G(2,2,s)\leq 3$ *provided* $s\geq 3$ *(recall from Corollary 3.6 that* $G(2,2^e,2)=4$*). In fact, extensive computer computations (not given here) suggest that*

$$G(2,2,s)=3 \ \text{ when } \ s\geq 3.$$

*The key point then comes from the fact that Theorem 3.17 below gives the sharper result* [2]

$$G(2,2^e,s)\leq 1 \ \text{ for } s\geq 3 \text{ and } e\geq 2.$$

**Proposition 3.16.** *The element in Eq. 3.24 is non-zero.*

*Proof.* The assertion for $s=3$ has been observed in the first sentence of Example 3.4. The case $s\geq 4$ then follows from (the proof of) Corollary 3.15. □

As anticipated in Example 3.4, we also describe, for $s\geq 3$ and $e\geq 2$, an almost sharp estimate for $\mathrm{TC}_s(\mathbb{F}(1,1,2^e))$.

[2]Computer calculations suggest that, in fact, $G(2,2^e,s)=1$ for $e\geq 2$ and $s\geq 3$.

**Theorem 3.17.** *For $e \geq 2$ and $s \geq 3$,*

$$0 \neq \left(z_{2,1}^{2^{e+1}-1} z_{2,2}^{2^{e+1}-1}\right) \cdot \left(z_{3,1}^{2^e+1} z_{3,2}^{2^e+3}\right) \cdot \left(z_{4,1}^{2^e+1} z_{4,2}^{2^e}\right) \cdots \left(z_{s,1}^{2^e+1} z_{s,2}^{2^e}\right)$$

*in $H^*(\mathbb{F}(1,1,2^e))^{\otimes s}$, consequently $s(2^{e+1}+1)-1 \leq \mathrm{TC}_s(\mathbb{F}(1,1,2^e)) \leq s(2^{e+1}+1)$.*

**Remark 3.9.** *Just as observed in Remark 3.8 in the case of Propositions 3.12 and 3.13, the powers of the factors $z_{2,j}$ in the product element of Theorem 3.17 coincide with the relevant powers of the product in Eq. 3.9 for $k = 2$. It is likely that such a phenomenon could shed light on possible generalizations of Theorems 3.11 and 3.17.*

**Remark 3.10.** *Theorem 3.17 fails for $e = 1$, as $z_{3,2}^4 = x_{1,2}^4 + x_{3,2}^4$, which vanishes in $H^*(\mathbb{F}(1,1,2))^{\otimes s}$ in view of Eq. 3.7.*

*Proof.* [Proof of Theorem 3.17] As in previous proofs, we can safely assume $s = 3$. Further, although we should not focus now on the top dimensional basis element of Eq. 3.20, the needed verifications are similar to those in the proof of Proposition 3.13. Indeed, this time we indicate how, for $e \geq 2$, the basis element $x_{1,1}^{2^e} x_{1,2}^{2^e} \cdot x_{2,1}^{2^e+1} x_{2,2}^{2^e} \cdot x_{3,1}^{2^e+1} x_{3,2}^{2^e}$ appears in the expression of $(z_{2,1}^{2^{e+1}-1} z_{2,2}^{2^{e+1}-1}) \cdot (z_{3,1}^{2^e+1} z_{3,2}^{2^e+3})$ in terms of the tensor basis in Eq. 3.19. The hypothesis $e \geq 2$ is used for the analysis of the mod-2 arithmetic of binomial coefficients. That being said, the calculation details can easily be carried out by the diligent reader. As a guide, we note that the three key steps are

$$\begin{aligned} z_{3,1}^{2^e+1} z_{3,2}^{2^e+3} &\equiv x_{1,1} x_{1,2}^2 + x_{1,2}^3\,, \\ z_{2,1}^{2^{e+1}-1} z_{2,2}^{2^{e+1}-1} &\equiv x_{1,1}^{2^e-1} x_{1,2}^{2^e-2} + x_{1,1}^{2^e-2} x_{1,2}^{2^e-1}, \end{aligned}$$

and the easy fact that $(x_{1,1}x_{1,2}^2 + x_{1,2}^3) \cdot (x_{1,1}^{2^e-1} x_{1,2}^{2^e-2} + x_{1,1}^{2^e-2} x_{1,2}^{2^e-1}) = x_{1,1}^{2^e} x_{1,2}^{2^e}$. ∎

**Remark 3.11.** *The results in this section suggest that purely cohomological methods could be used to give, for positive integers $i$ and $k$, an estimate of the higher topological complexity of $\mathbb{F}(1^k, 2^e - k + i)$ giving $G(k, 2^e - k + i, s) < i$ provided $e$ is sufficiently large. An interesting additional restriction of the form $k + i - 1 \leq s$, which would be compatible with the corresponding restrictions in Theorems 3.11 and 3.17 (as well as with computer calculations not shown here), might also be needed.*

Chapter 4

# Projective Product Coverings and Sequential Motion Planning Algorithms in Real Projective Spaces

## 4.1 The Projective Product Covering

A main goal in this thesis is to generalize a classic result from Farber, Yuzvinsky and Tabachnikov about the relationship between $\mathrm{TC}(\mathbb{R}\mathrm{P}^n)$ and $\mathrm{Imm}(\mathbb{R}\mathrm{P}^n)$. In section 2.5 we reviewed the principal results of [15], in order to generalize the equality

$$\mathrm{TC}(\mathbb{R}\mathrm{P}^n) = \mathrm{Imm}(\mathbb{R}\mathrm{P}^n).$$

Which holds for $n \neq 1, 3, 7$.

The calculation of immersion dimension for $\mathbb{R}\mathrm{P}^n$ is an open problem; despite it is well known in many cases no general formula is known to work for $n$.

In section 2.2 we defined higher topological complexity. Recall that for $s \geq 2$ the *s-th higher topological complexity* of a path connected space $X$ ($\mathrm{TC}_s(X)$), is defined as the reduced Schwarz genus of the fibration:

$$\begin{aligned} e_s &= e_s^X : X^{[0,1]} \to X^s; \\ e_s(\gamma) &= \left(\gamma\left(\frac{0}{s-1}\right), \gamma\left(\frac{1}{s-1}\right), \ldots, \gamma\left(\frac{s-1}{s-1}\right)\right). \end{aligned}$$

Thus, $\mathrm{TC}_s(X)+1$ is the smallest cardinality of open covers $\{U_i\}_i$ of $X^s$ so that $e_s$ admits a continuous section $\sigma_i$ on each $U_i$. The open sets $U_i$ of such an open covering are called *s-local domains*. Corresponding sections $\sigma_i$ are said to be the *s-local rules*, and the resulting family of pairs $\{(U_i, \sigma_i)\}$ is called an *s-motion planning algorithm* for $X$. We say that such a family is an *optimal s-motion planning algorithm* if it has $\mathrm{TC}_s(X)+1$ $s$-local domains.

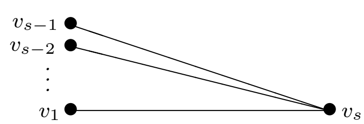
If we have $s$-states in a robot with configuration space $X$, we may think that $\mathrm{TC}_s(X)$ gives instructions to do $s$-actions at same time, that is, if we have $A_1, \cdots, A_s$ states from a robot and the previous $\{(U_i, \sigma_i)\}$, then the algorithm sets as follows (Algorithm 3).

**Algorithm 3:** Higher Motion Planning Algorithm

**Data:** $s_i, U_i$ for any $i = 1, \ldots, k$
**Result:** $s_i(A_1, \cdots, A_s)$
**begin**
  $(A_1, \cdots, A_s) \longleftarrow$ chosen ;
  $i \longleftarrow$ smallest s.t. $(A_1, \cdots, A_s) \in U_i$;
**end**
**return** $s_i(A_1, \cdots, A_s)$

These ideas generalize the concept of topological complexity, introduced by Farber in [12], as a model to study the continuity instabilities in the motion planning of an autonomous systems (robots), whose space of configurations is $X$. The term *higher* arises by considering the base space $X^s$ of $e_s$ as a serial of prescribed stages in the robot motion planning, while Farber's original case $s = 2$, deals only with the space $X \times X$ of initial-final stages.

**Remark 4.1.** *As we saw in the considerations following Definition 2.4,* $\mathrm{TC}_s(X)$ *can equivalently be defined as the genus of the evaluation map* $X^{\Gamma_s} \to X^s$, $\gamma \mapsto (\gamma(v_1), \ldots, \gamma(v_s))$, *where* $\Gamma_s$ *is (the underlying topological space of) a given connected graph, and* $v_1, \ldots, v_s$ *are* $s$ *distinct vertices of* $\Gamma_s$. *In the final section of this chapter, it will be convenient to take* $\Gamma_s$ *to be the graph with exactly* $s$ *vertices* $v_1, v_2, \ldots v_s$, *and* $s-1$ *edges* $(v_1, v_s), (v_2, v_s), \ldots (v_{s-1}, v_s)$ *depicted as follows:*




Most of the existing methods to estimate the higher topological complexity of a space, are cohomological in nature. One of the most successful such methods is a special case of Proposition 4.1 below, which is easily proved on the lines of [32, Theorem 4 in page 73].

**Proposition 4.1.** *Let $h^*$ be a generalized cohomology theory with products. The sectional category of a fibration $\pi\colon E \to B$ is no less than the cup length of elements in the kernel of $\pi^*\colon h^*(B) \to h^*(E)$.*

Here *cup-length* refers to the maximal number of elements in the indicated ideal having a non-vanishing product. Later in this chapter, we will apply Proposition 4.1 to the $(\mathbb{Z}_2)^{s-1}$-covering space $\pi_s$ in (1.1). The covering space is explicitly defined and studied in this section. Let the group $(\mathbb{Z}_2)^{s-1}$, with obvious generators $\sigma_i$ $(1 \le i \le s-1)$, act on $(S^m)^{\times s}$ so that

$$\sigma_i \cdot (x_1, \ldots, x_s) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_s). \tag{4.1}$$

Let $\mathrm{P}_{\mathbf{m}_\mathrm{s}}$ be the quotient of $(S^m)^{\times s}$ by the involution $\delta \cdot (x_1, \ldots, x_s) = (-x_1, \ldots, -x_s)$. It is elementary to check that the induced $(\mathbb{Z}_2)^{s-1}$-action on $\mathrm{P}_{\mathbf{m}_\mathrm{s}}$ is principal and has orbit space $(\mathbb{R}\mathrm{P}^m)^{\times s}$. This defines the $(\mathbb{Z}_2)^{s-1}$-principal bundle $\pi_s$.

For a path $\gamma$ in $\mathbb{R}\mathrm{P}^m$, pick a lifting $\widetilde{\gamma}$ through the projection $S^m \to \mathbb{R}\mathrm{P}^m$, and note that the class of

$$\left(\widetilde{\gamma}\left(\frac{0}{s-1}\right), \widetilde{\gamma}\left(\frac{1}{s-1}\right), \ldots, \widetilde{\gamma}\left(\frac{s-1}{s-1}\right)\right)$$

in $\mathrm{P}_{\mathbf{m}_\mathrm{s}}$ does not depend on the chosen lifting $\widetilde{\gamma}$. We get a map $(\mathbb{R}\mathrm{P}^m)^{[0,1]} \to \mathrm{P}_{\mathbf{m}_\mathrm{s}}$ fitting in the commutative diagram

$$\begin{tikzcd} (\mathbb{R}\mathrm{P}^m)^{[0,1]} \arrow[rr] \arrow[dr, "e_s"'] & & \mathrm{P}_{\mathbf{m}_\mathrm{s}} \arrow[dl, "\pi_s"] \\ & (\mathbb{R}\mathrm{P}^m)^{\times s}, & \end{tikzcd} \tag{4.2}$$

which readily yields (1.1).

The homotopy nature of $\pi_s$ is described through its classifying map as:

**Proposition 4.2.** *For $1 \le i \le s$ let $p_i\colon (\mathbb{R}\mathrm{P}^m)^{\times s} \to \mathbb{R}\mathrm{P}^m$ be the $i$-th projection, $\xi_m \to \mathbb{R}\mathrm{P}^m$ be the Hopf bundle over $\mathbb{R}\mathrm{P}^m$, and $\mu_s\colon (\mathbb{R}\mathrm{P}^m)^{\times s} \to (\mathbb{R}\mathrm{P}^\infty)^{\times(s-1)}$ classify $\pi_s$. Then, for $1 \le i \le s-1$, the $i$-th component $\mu_{i,s}$ of $\mu_s$ classifies $p_i^*(\xi_m) \otimes p_s^*(\xi_m)$.*

The conclusion of Proposition 4.2 can of course be stated by saying that $\mu_{i,s}$ is homotopic to the composition of the projection $p_{i,s}\colon (\mathbb{R}\mathrm{P}^m)^{\times s} \to \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ onto the $(i,s)$ coordinates, the inclusion $\mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \hookrightarrow \mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty$, and the Hopf multiplication $\mu\colon \mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty \to \mathbb{R}\mathrm{P}^\infty$.

*Proof.* [Proof of Proposition 4.2] Recall $\delta$ stands for the involution

$$(x_1, \dots, x_s) \mapsto (-x_1, \dots, -x_s)$$

in $(S^m)^{\times s}$ so, by definition, the corresponding orbit space is $\mathrm{P}_{\mathbf{m}_s}$. The total space $Z_i$ of the $\mathbb{Z}_2$-principal bundle classified by the $i$-th component $\mu_{i,s}$ is the quotient of $(S^m)^{\times s}$ by the actions of $\delta$ and of those $\sigma_\ell$ $(1 \le \ell \le s-1)$ with $\ell \neq i$, and where the $\mathbb{Z}_2$-principal action on $Z_i$ is induced by change of signs on the $i$-th coordinate. Let $\lambda_{i,j} \to \mathbb{R}\mathrm{P}^m$ stand for the restriction to the $j$-th axis of the latter double covering (axes are taken with respect to the base point in $\mathbb{R}\mathrm{P}^m$ given by the class of $1 := (1, 0, \dots, 0) \in S^m$).

**Case** $j = s$: Note that a class in $\lambda_{i,s}$ has a unique representative of the form $(1, \dots, 1, x_s)$ and, in these terms, the $\mathbb{Z}_2$-principal action on $\lambda_{i,s}$ is given by

$$\begin{aligned} \Big[(1, \dots, 1, x_s)\Big] \mapsto \Big[(1, \dots, 1, \overset{i}{-1}, 1, \dots, 1, x_s)\Big] &= \Big[(-1, \dots, -1, \overset{i}{1}, -1, \dots, -1, -x_s)\Big] \\ &= \Big[(1, \dots, 1, \overset{i}{1}, 1, \dots, 1, -x_s)\Big], \end{aligned}$$

where the notation $\overset{b}{a}$ indicates that the number $a$ appears in the $b$-th coordinate of the $s$-tuple. Consequently, $\lambda_{i,s} \to \mathbb{R}\mathrm{P}^m$ is homeomorphic to the Hopf projection $S^m \to \mathbb{R}\mathrm{P}^m$.

**Case** $j = i$: As above, a class in $\lambda_{i,i}$ has a unique representative of the form

$$(1, \dots, 1, \overset{i}{x_i}, 1, \dots, 1)$$

and, now, the $\mathbb{Z}_2$-principal action on $\lambda_{i,i}$ is antipodal on $x_i$ on the nose. Thus $\lambda_{i,i} \to \mathbb{R}\mathrm{P}^m$ is also homeomorphic to the Hopf projection $S^m \to \mathbb{R}\mathrm{P}^m$.

**Case** $j \notin \{i, s\}$: Classes in $\lambda_{i,j}$ are represented by elements

$$(\pm 1, \dots, \pm 1, \overset{j}{x_j}, \pm 1, \dots, \pm 1, \overset{i}{\pm 1}, \pm 1, \dots, \pm 1)$$

where, to fix ideas, we have assumed $j < i < s$ —the case $i < j < s$ works just as well. Dividing out first by the action of $\delta$ and of the $\sigma_\ell$ with $\ell \notin \{i, j\}$ (and then by the action of $\sigma_j$), we see that $\lambda_{i,j}$ is given as the quotient of $S^m \times \mathbb{Z}_2$ by the antipodal action on the first coordinate and with $\mathbb{Z}_2$-principal action coming from the antipodal action on the second coordinate. In other words, $\lambda_{i,j} \to \mathbb{R}\mathrm{P}^m$ is the trivial $\mathbb{Z}_2$-bundle.

The conclusion follows. $\square$

## 4.2 Motion Planning Algorithms through Equivariant Maps

Recall that the $(k+1)$-iterated self join-power of a topological space $X$, $J_k(X)$, is defined inductively by $J_k(X) := J_{k-1}(X) * X$ $(k \geq 1)$ where $J_0(X) = X$. Then, for a topological group $G$, $B_kG := J_k(G)/G$ is the $k$-th stage in Milnor's construction of the classifying space $BG := J_\infty(G)/\,G$, where $G$ acts diagonally on the vertices of $J_\infty(G) := \bigcup_{k\geq 0} J_k(G)$ —so barycentric coordinates are preserved.

In what follows $G_s$ stands for the (discrete) group $(\mathbb{Z}_2)^{\times(s-1)}$. By [32, Theorem 9 in page 86], the classifying homotopy class $\mu_s$ in Proposition 4.2 has a representative factoring in the form

$$(\mathbb{R}\mathrm{P}^m)^{\times s} \xrightarrow{\beta_s} B_{\mathrm{secat}(\pi_s)}(G_s) \hookrightarrow B(G_s) \simeq (\mathbb{R}\mathrm{P}^\infty)^{\times(s-1)}, \tag{4.3}$$

where $\beta_s$ is covered by a $G_s$-equivariant map $\alpha_s\colon \mathrm{P}_{\mathbf{m}_\mathrm{s}} \to J_{\mathrm{secat}(\pi_s)}(G_s)$. Then, in terms of the $G_s$-action defined in Eq.4.1, the composition of the canonical projection $(S^m)^{\times s} \to \mathrm{P}_{\mathbf{m}_\mathrm{s}}$ with $\alpha_s$ yields a $G_s$-equivariant map $\phi_s\colon (S^m)^{\times s} \to J_{\mathrm{secat}(\pi_s)}(G_s)$ satisfying the condition

$$\phi_s(x_1,\ldots,x_{s-1},-x_s) = \sigma_1\cdots\sigma_{s-1}\cdot\phi_s(x_1,\ldots,x_{s-1},x_s), \text{ for all } (x_1,\cdots,x_s)\in(S^m)^{\times s}. \tag{4.4}$$

**Conjecture 4.1.** *An $s$-motion planning algorithm for $\mathbb{R}\mathrm{P}^m$ with $\mathrm{secat}(\pi_s)+1$ $s$-local rules can be constructed out of a map $\phi_s$ as above. Consequently $\mathrm{secat}(\pi_s) \geq \mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)$, and Eq. 1.1 becomes an equality for any $s \geq 2$.*

The conjecture is motivated in part by (the proof of) Proposition 2.25, which asserts that the case $s = 2$ of Conjecture 4.1 holds true —see Proposition 4.4 and Remark 4.3 below. Corollary 4.9 in the next section is meant to gather further evidence for the plausibility of Conjecture 4.1.

**Remark 4.2.** *One of our main interests in Conjecture 4.1 is the possibility of obtaining upper bounds for $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)$ from the construction of $G_s$-equivariant maps $\phi_s\colon (S^m)^{\times s} \to J_k(G_s)$ satisfying Eq. 4.4. Indeed, such a map covers a map $\beta_s$ as in Eq. 4.3, so that [32, Theorem 9 in page 86] implies $k \geq \mathrm{secat}(\pi_s)$, and so $k \geq \mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)$ if Conjecture 4.1 were to hold.*

Given spaces $X$ and $Y$, consider the open subspace $U \subset X * Y$ consisting of the barycentric expressions $t_0x + t_1y$ with ($x \in X$, $y \in Y$, $0 \leq t_i$, $t_0 + t_1 = 1$, and) $t_1 > 0$. Observe that, if $Y$ is discrete, $U$ is a topological disjoint union of open cones with base $X$ (the cones are open in the sense that they are missing their base). In such terms, the following auxiliary result becomes self-evident.

**Lemma 4.3.** *For $k \geq 0$, $s \geq 2$, and $0 \leq j \leq k$, consider the open set $U_j \subset J_k(G_s)$ consisting of the barycentric expressions $\sum_{\ell=0}^{k} t_\ell g_\ell$ with $t_j > 0$ (here, as usual, $g_\ell \in G_s$, $t_\ell \geq 0$, and $\sum t_\ell = 1$). Then $U_j$ is closed under the action of $G_s$, and has $2^{s-1}$ connected components, each of which is open in $U_j$ and contractible (in itself). Further, the induced $G_s$-action on the set of connected components of $U_j$ has a single orbit.*

**Proposition 4.4.** *Let $D_s = \{(x_1, \ldots, x_s) \in (S^m)^{\times s} \colon x_i = x_s \text{ for some } i \in \{1, \ldots, s-1\}\}$. The conclusions in Conjecture 4.1 hold true if one starts with a $G_s$-equivariant map $\phi_s \colon (S^m)^{\times s} \to J_{\mathrm{secat}(\pi_s)}(G_s)$ satisfying Eq. 4.4 together with one of the following conditions:*

1. *For* every $j \in \{0, 1, \ldots, \mathrm{secat}(\pi_s)\}$, *$\phi_s(D_s)$ intersects at most a single component of $U_j$.*
2. *For* some $j_0 \in \{0, 1, \ldots, \mathrm{secat}(\pi_s)\}$, *$\phi_s(D_s)$ is fully contained in some component of $U_{j_0}$.*

**Remark 4.3.** *The easy fact that, for $s = 2$, there exist maps $\phi_2$ as that assumed in Proposition 4.4 was first noted in [15, Lemmas 5.3 and 5.7]. Explicitly, it is standard that the case $m = 1, 3, 7$ can be accounted by using the multiplication in the complex, quaternion, and octonion numbers, respectively. For $m \neq 1, 3, 7$, since the diagonal inclusion $\mathbb{R}\mathrm{P}^m \hookrightarrow \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ is a cofibration, any axial map $\alpha \colon \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^{\mathrm{secat}(\pi_2)}$, being nullhomotopic on the diagonal[1], is homotopic to a map $\alpha' \colon \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^{\mathrm{secat}(\pi_2)}$ which is (necessarily axial and) actually constant on the diagonal. Then any map $\phi_2 \colon S^m \times S^m \to J_{\mathrm{secat}(\pi_2)}(\mathbb{Z}_2) = S^{\mathrm{secat}(\pi_2)}$ covering $\alpha'$ is a fortiori constant on the diagonal. In particular, such maps $\phi_2$ satisfy* both *conditions (1) and (2) in Proposition 4.4 for, obviously, the singleton $\phi_2(D_2)$ is fully contained in some component of* each *$U_j$ satisfying $\phi_2(D_2) \cap U_j \neq \varnothing$.*

*Proof.* [Proof of Proposition 4.4] For $0 \leq j \leq \mathrm{secat}(\pi_s)$, set $V_j = \phi_s^{-1}(U_j) \subseteq (S^m)^{\times s}$, and $W_j = q(V_j) \subseteq (\mathbb{R}\mathrm{P}^m)^{\times s}$ where $q$ stands for the composition $(S^m)^{\times s} \to \mathrm{P}_{\mathbf{m}_s} \to (\mathbb{R}\mathrm{P}^m)^s$ of canonical projections. Note that the equality

$$V_j = q^{-1}(W_j) \tag{4.5}$$

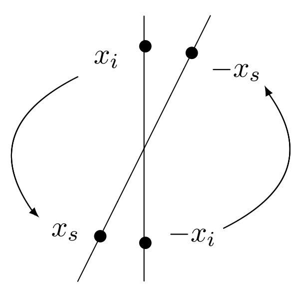
[1] This uses the fact that $\mathrm{secat}(\pi_2) > m$, which in turn comes from the assumption $m \neq 1, 3, 7$ (compare to Remark 4.4).

holds since $V_j$ is closed under the action of $\delta$ and of the $\sigma_\ell$ with $1 \leq \ell \leq s-1$ (as the $G_s$-equivariant map $\phi_s$ satisfies Eq. 4.4). Further, the sets $W_0, \ldots, W_{\mathrm{secat}(\pi_s)}$ form an open cover of $(\mathbb{R}\mathrm{P}^m)^{\times s}$.

If condition (1) in the statement of the Proposition holds, we complete the proof by constructing local sections $\varsigma_j\colon W_j \to (\mathbb{R}\mathrm{P}^m)^{\Gamma_s}$ $(0 \leq j \leq \mathrm{secat}(\pi_s))$ for the evaluation map $(\mathbb{R}\mathrm{P}^m)^{\Gamma_s} \to (\mathbb{R}\mathrm{P}^m)^{\times s}$ described at the end of Remark 4.1. Details follow. Fix $j \in \{0, 1, \ldots, \mathrm{secat}(\pi_s)\}$. If $\phi_s(D_s)$ intersects $U_j$, let $U_{j,0}$ denote the component of $U_j$ containing $\phi_s(D_s) \cap U_j$; otherwise, choose any component $U_{j,0}$ of $U_j$. For $(L_1, \ldots, L_s) \in W_j$, the $2^s$ elements in $q^{-1}\{(L_1, \ldots, L_s)\}$ lie in $V_j$, in view of Eq. 4.5. Also, in view of Eq. 4.4 and the final assertion in Lemma 4.3, exactly two elements in $q^{-1}(L_1, \ldots, L_s)$ have $\phi_s$-image in $U_{j,0}$. Indeed, if one of the latter elements is $(x_1, x_2, \ldots, x_s)$, then the other is $(-x_1, -x_2, \ldots, -x_s)$. Furthermore, in these conditions,

$$\text{if } L_i = L_s \text{ for some } 1 \leq i < s, \text{ then in fact } x_i = x_s, \tag{4.6}$$

by construction (and in view of Lemma 4.3). Then $\varsigma_j(L_1, \ldots, L_s)\colon \Gamma_s \to \mathbb{R}\mathrm{P}^m$ is defined to be the map whose restriction to the (oriented) edge $(v_i, v_s)$ describes the uniform-speed motion in $\mathbb{R}\mathrm{P}^m$ from $L_i$ to $L_s$ obtained by rotating $L_i$ toward $L_s$ along the plane generated by these two lines (no motion if $L_i = L_s$), and in such a way that the corresponding rotation from $x_i$ to $x_s$ is performed through an angle smaller than 180 degrees. As shown in the picture below, the latter requirement holds independently of whether one uses $(x_1, \ldots, x_s)$ or $(-x_1, \ldots, -x_s)$, so that $\varsigma_j(L_1, \ldots, L_s)$ is well defined.




The resulting function $\varsigma_j$ is clearly a section on $W_j$ for the evaluation map $(\mathbb{R}\mathrm{P}^m)^{\Gamma_s} \to (\mathbb{R}\mathrm{P}^m)^s$. Lastly, the continuity of $\varsigma_j$ follows from Eq. 4.6, and from the facts that $U_{j,0}$ is open, that $\phi_s$ is continuous, and that $q$ is a covering projection.

A minor modification of the above construction is needed in order to complete the proof when condition (2) in the statement of the Proposition holds. Indeed, in the notation above, the problematic $q(D_s)$ is contained in $W'_{j_0} := W_{j_0}$, while condition (2) assures that the construction above yields the needed local section $\zeta'_{j_0} = \zeta_{j_0}\colon W'_{j_0} \to (\mathbb{R}\mathrm{P}^m)^{\Gamma_s}$. For all other $j \neq j_0$ we set $W'_j := W_j - q(D_s)$ (so that the sets $W'_i$ with $0 \leq i \leq \mathrm{secat}(\pi_s)$ cover $(\mathbb{R}\mathrm{P}^m)^{\times s}$), which (is open and) vacuously avoids the possibility of the failure of Eq. 4.6, thus yielding an obviously continuous local section $\zeta'_j\colon W'_j \to (\mathbb{R}\mathrm{P}^m)^{\Gamma_s}$. □

Regarding a potential proof of Conjecture 4.1, it is possible that, for general $s \geq 2$, Proposition 4.2 would have to play a key role in proving the existence of a map $\phi_s$ as the one assumed in Proposition 4.4. However, the problem seems to be much more subtle for $s \geq 3$ than the rather straightforward instance $s = 2$. We close this section by pinpointing some of the intricacies that are inherent to a potential proof of Conjecture 4.1 via Proposition 4.2 when $s \geq 3$, and how this leads to a couple of interesting new challenges in the field (which we hope to address elsewhere).

**Remark 4.4.** *We start by discussing the relevance of the inequality*

$$\mathrm{secat}(\pi_s\colon \mathrm{P}_{\mathbf{m}_s} \to (\mathbb{R}\mathrm{P}^m)^{\times s}) \geq (s-1)m, \tag{4.7}$$

*with strict inequality if $m+1$ is not a power of 2 (obtained in Eq.4.8 and Remark 4.6 below) in a potential proof of Conjecture 4.1. Recall that the isomorphism class of the $G_s$-principal bundle $\pi_s$ has been described in Proposition 4.2 via the homotopy type of its classifying map $\mu_s\colon (\mathbb{R}\mathrm{P}^m)^s \to (\mathbb{R}\mathrm{P}^\infty)^{\times(s-1)}$. Of course, the homotopy type of any map $\beta_s\colon (\mathbb{R}\mathrm{P}^m)^s \to B_{\mathrm{secat}(\pi_s)}(G_s)$ fitting in the factorization Eq.4.3 does not have to be determined by that of $\mu_s$. Nonetheless, as noted in Remark 4.3, a key fact in the proof of the $s=2$ case of Conjecture 4.1 is that any such $\beta_2$ remains being null homotopic on the diagonal when $m \neq 1, 3, 7$, as* $\mathrm{secat}(\pi_2) > m$ *for those values of $m$. (As explained in [15, Lemma 5.4], the latter inequality turns out to be closely related to Adams' solution of the Hopf invariant 1 problem.) Now, for $s \geq 3$, the diagonal is replaced by the "pivoted" diagonal $q(D_s)$ used at the end of the proof of Proposition 4.4. Then, in order to understand the homotopy properties of the restricted $\beta_s|_{D_s}$ from the corresponding properties of the restricted $\mu_s|_{D_s}$ (as in the case $s = 2$), we would need to know that* $\dim(D_s)$ *is* strictly smaller *than the connectivity of the inclusion $B_{\mathrm{secat}(\pi_s)}(G_s) \hookrightarrow B_\infty(G_s) \simeq (\mathbb{R}\mathrm{P}^\infty)^{\times(s-1)}$. Such a condition is assured by Eq.4.7 if $m+1$ is not a power of 2, as the latter map is a* $\mathrm{secat}(\pi_s)$*-equivalence (its homotopy fiber agrees with that for the (obviously)* $\mathrm{secat}(\pi_s)$*-equivalence $J_{\mathrm{secat}(\pi_s)} \to *$), while $D_s$ is a union of subcomplexes of $(\mathbb{R}\mathrm{P}^m)^{\times s}$ each homeomorphic to $(\mathbb{R}\mathrm{P}^m)^{\times(s-1)}$, so that* $\dim(D_s) = (s-1)m$*. Consequently, the first task to deal with in a proof of Conjecture 4.1 based on Proposition 4.4 is to decide whether Eq.4.7 can be improved to a strict inequality when $m+1$ is a power* *of 2. As indicated in Example 4.1 below, Eq.4.7 is in fact an equality for $m = 1, 3, 7$, in which case Eq.1.1 is an equality too. Thus, the real initial task is to decide whether Eq.4.7 actually improves to a strict inequality for $m = 2^e - 1$ with $e \geq 4$ —just as in the case $s = 2$. A particularly interesting feature of such a challenge is to understand how a potential strict inequality in Eq.4.7 would fit within (a possibly generalized form of) the Hopf invariant 1 problem.*

**Remark 4.5.** *In addition to the considerations in Remark 4.4, it should be noted that, unlike the situation for $s = 2$, no map $\beta_s$ as above can be nullhomotopic on $D_s$ when $s \geq 3$ for, in fact, $\mu_s$ evidently fails to be nullhomotopic on $D_s$. Consequently, unlike the situation for $s = 2$ discussed in Remark 4.3, the issue of being able to "fix" a $G_s$-equivariant map $\phi_s$ as in Eq.4.4 so to satisfy at least one of the two conditions in Proposition 4.4 requires handling non-trivial homotopy information.*

## 4.3 Cohomology Estimates

This section is devoted to estimating the sharpness of Eq.1.1 by means of cohomological methods. In particular, we show equality for all even $m$ when $s$ is large enough. Explicitly, an application of Proposition 4.1 to $e_s$, which is a fibrational replacement for the diagonal $\Delta_s\colon X \hookrightarrow X^{\times s}$, yields the lower bound

$$\mathrm{TC}_s(X) \geq \mathrm{zcl}_s^{h^*}(X),$$

where $\mathrm{zcl}_s^{h^*}(X)$ is the $h^*$-cup-length of $s$-th zero-divisors in $X$, i.e. of elements in the kernel of the induced map $\Delta_s^*\colon h^*(X^{\times s}) \to h^*(X)$ (see [2, Definition 3.8]). In this section we show that, when $X := \mathbb{R}\mathrm{P}^m$ and $h^* := H^*$ is singular cohomology with mod 2 coefficients, $\mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m) := \mathrm{zcl}_s^{H^*}(\mathbb{R}\mathrm{P}^m)$ is in fact a lower bound for the right hand-side in (1.1), which, for $m$ odd and $s$ large enough, agrees with the well known upper bound $sm \geq \mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)$ coming from [2, Theorem 3.9].

Recall that $H^*((\mathbb{R}\mathrm{P}^m)^{\times s}) = H^*(\mathbb{R}\mathrm{P}^m)^{\otimes s}$ is the $\mathbb{Z}_2$-algebra generated by the classes $x_i = p_i^*(x)$ subject to the relations $x_i^{m+1} = 0$, $1 \leq i \leq s$, where $x \in H^1(\mathbb{R}\mathrm{P}^m)$ is the first Stiefel-Whitney class of $\xi_m$, and $p_i$ is defined in Proposition 4.2. We do not stress the dependence of $x_i$ on $s$ because, if $s' > s$ and $\pi_{s,s'}\colon (\mathbb{R}\mathrm{P}^m)^{\times s'} \to (\mathbb{R}\mathrm{P}^m)^{\times s}$ is the projection onto the first $s$ coordinates, then we think of the map induced in cohomology by $\pi_{s,s'}$ as a honest inclusion. The standard (graded) basis of $H^*((\mathbb{R}\mathrm{P}^m)^{\times s})$ consists of all the monomials $x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s}$ with

$0 \leq a_i \leq m$. Note that each $x_i + x_s$ $(1 \leq i \leq s-1)$ is an $s$-th zero-divisor, so it pulls back trivially under the evaluation map $e_s^*$. In fact:

**Proposition 4.5.** *For $1 \leq i \leq s-1$, $x_i + x_s$ pulls back trivially under the map $\pi_s$ on the right of Eq.4.2.*

*Proof.* The projection $(S^m)^{\times s} \to S^m \times S^m$ onto the $(i,s)$ axes induces a map from $\pi_s$ to $\pi_2$ lying over $p_{i,s}$. The conclusion then follows since $x \otimes 1 + 1 \otimes x \in H^1(\mathbb{R}\mathrm{P}^m \times RP^m)$, the mod 2 Euler class of the exterior product $\xi_m \otimes \xi_m$, vanishes under $\pi_2$, which is the sphere bundle of $\xi_m \otimes \xi_m$. □

**Lemma 4.6.** *The ideal of $s$-th zero-divisors in $H^*(\mathbb{R}\mathrm{P}^m)^{\otimes s}$ is generated by the elements $x_i + x_s$ in Proposition 4.5.*

*Proof.* Let $\sum_{(a_1,\ldots,a_s)} x_1^{a_1} \cdots x_s^{a_s}$ be the expression of an homogeneous $s$-th zero-divisor $z$ in terms of the standard basis. Note that the number of summands must be even if $\deg(z) \leq m$. Thus, it suffices to prove that the following elements lie in the ideal $I_s$ generated by the binomials $x_i + x_s$:

(1) The sum of any two basis elements in degree at most $m$.

(2) A basis element in degree greater than $m$.

Elements in (1) are easily dealt with by induction on the degree and on the number of common factors. For instance

$$x_1x_2 + x_3x_4 = (x_1x_2 + x_2x_3) + (x_2x_3 + x_3x_4) = x_2(x_1 + x_3) + x_3(x_2 + x_4).$$

Elements in (2) are dealt with also by an inductive argument based on the fact that, for $i < j$,

$$\begin{aligned} x_i^{a_i} x_{i+1}^{a_{i+1}} \cdots x_j^{a_j} &= (x_i + x_j) \cdot x_i^{a_i - 1} x_{i+1}^{a_{i+1}} \cdots x_j^{a_j} + x_i^{a_i - 1} x_{i+1}^{a_{i+1}} \cdots x_{j-1}^{a_{j-1}} x_j^{a_j + 1} \\ &\equiv x_i^{a_i - 1} x_{i+1}^{a_{i+1}} \cdots x_{j-1}^{a_{j-1}} x_j^{a_j + 1}, \end{aligned}$$

where the congruence holds module $I_s$. □

Thus (1.1) extends to

$$sm \geq \mathrm{TC}_s(\mathbb{R}\mathrm{P}^m) \geq \mathrm{secat}(\pi_s) \geq \mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m). \tag{4.8}$$

Set $G(m,s) = sm - \mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m)$, so equality holds in (1.1) whenever $G(m,s) = 0$.

**Lemma 4.7.** *$G(m,2) \geq G(m,3) \geq G(m,4) \geq \cdots \geq 0$.*

*Proof.* Thinking in terms of the expression of elements as sums of the standard basis of $H^*(\mathbb{R}\mathrm{P}^m)^{\otimes s}$, we see that if $z \in H^*((\mathbb{R}\mathrm{P}^m)^{\times s})$ is a non-zero product of $s$-th zero-divisors, then

$$z \cdot (x_1 + x_{s+1})^m = z \cdot (x_{s+1}^m + \cdots) \neq 0.$$

So, $\mathrm{zcl}_{s+1}(\mathbb{R}\mathrm{P}^m) \geq \mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m) + m$ and the result follows. $\square$

**Remark 4.6.** *It is elementary to check that $\mathrm{zcl}_2(\mathbb{R}\mathrm{P}^m) = 2^{z(m)} - 1$, where $z(m)$ is the integral part of $\log_2(2m)$ (c.f. [15, Theorem 4.5]). The last line in the previous proof then implies $\mathrm{zcl}_s(\mathbb{R}\mathrm{P}^m) \geq (s-1)m$ with strict inequality if $m+1$ is not a power of 2. The proof of Theorem 4.8 below is based on a streamlined version of the previous inequality.*

The monotonic sequence of non-negative integers in Lemma 4.7 stabilizes, and we denote by $G(m)$ the corresponding stable value.

**Theorem 4.8.** *Assume $m \equiv 2^e - 1 \bmod 2^{e+1}$ with $e \geq 0$. In other words, $e$ is the length of the block of consecutive ones ending the binary expansion of $m$. For instance, $e = 0$ if and only if $m$ is even. Then $G(m) \leq 2^e - 1$ with equality if $m$ is even, or if $m = 2^e - 1$. In fact, $G(m,s) \leq 2^e - 1$ for $s \geq \max\{(m+1)/2^e, 2\}$. Specifically, if $m > 2^e - 1$ and $\sigma$ stands for $(m+1)/2^e$ (so $\sigma$ is an integer greater than 2), then the product of $\sigma$-th zero-divisors*

$$(x_1 + x_\sigma)^{m+2^e} \cdots (x_{\sigma-1} + x_\sigma)^{m+2^e} \in H^*((\mathbb{R}\mathrm{P}^m)^\sigma)$$

*is non-zero.*

**Conjecture 4.2.** *In Theorem 4.8, the equality*[2] *$G(m) = 2^e - 1$ holds without restriction on $e$.*

Example **4.1.** *For $e \geq 1$ and $s \geq 2$,*

$$\begin{aligned} 0 &\neq x_1^{2^e-1} x_2^{2^e-1} \cdots x_{s-1}^{2^e-1} + \cdots \\ &= (x_1 + x_s)^{2^e-1} (x_2 + x_s)^{2^e-1} \cdots (x_{s-1} + x_s)^{2^e-1} \\ &\in H^*((\mathbb{R}\mathrm{P}^{2^e-1})^{\times s}), \end{aligned}$$

[2] Conjecture 4.2 has recently been proved in [4, Theorem 3.3].

*which yields $G(2^e-1,s)\leq 2^e-1$. The latter inequality is in fact an equality in view of Lemma 4.6 and the fact that the $2^e$-th power of any element in $H^*((\mathbb{R}\mathrm{P}^{2^e-1})^{\otimes s})$ vanishes. In the case of the three Hopf spaces $\mathbb{R}\mathrm{P}^1$, $\mathbb{R}\mathrm{P}^3$, and $\mathbb{R}\mathrm{P}^7$, the $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)$-contents of the assertion $G(2^e-1,s)=2^e-1$ is strengthened by [28, Theorem 1]: $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)=m(s-1)$ for all $s$ and $m\in\{1,3,7\}$. Thus, for all $s\geq 2$ and $m\in\{1,3,7\}$, the last three terms in Eq. 4.8 are all equal to $m(s-1)$.*

*Proof.* [Proof of Theorem 4.8] The case $m=2^e-1$ is accounted for in Example 4.1, so we assume $m>2^e-1$. The hypothesis on $m$ and $e$ implies that the binomial coefficient $\binom{m+2^e}{2^e}$ is odd, so

$$(x_i+x_\sigma)^{m+2^e}=x_i^m x_\sigma^{2^e}+\textit{terms involving powers } x_i^j \textit{ with } j<m$$

for $2\leq i\leq\sigma$. Therefore, ignoring basis elements $x_1^{a_1}\cdots x_\sigma^{a_\sigma}$ having $a_i<m$ for some $i\in\{1,\ldots,\sigma-1\}$, the product of $\sigma$-th zero-divisors under consideration becomes

$$(x_1^m x_\sigma^{2^e})(x_2^m x_\sigma^{2^e})\cdots(x_{\sigma-1}^m x_\sigma^{2^e})=x_1^m x_2^m\cdots x_{\sigma-1}^m x_\sigma^{(\sigma-1)2^e},$$

which is an element of the standard basis. □

Corollary 4.9 below, a direct consequence of Theorem 4.8, should be compared with the final assertion in Example 4.1.

**Corollary 4.9.** *If $m$ is even and $s>m$, all inequalities in Eq. 4.8 are in fact equalities.*

The hypothesis $s>m$ can substantially be relaxed in many cases. For instance, [17, Theorem 1.2] implies that the conclusion in Corollary 4.9 remains true for all $s\geq 3$ if $m$ is a 2-power. Other concrete instances follow from Propositions 4.2, 4.7 and 4.9–4.12 in [4].

## 4.4 Examples with $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^m)=\mathrm{secat}(\pi_s\colon \mathrm{P}_{\mathbf{m}_s}\to(\mathbb{R}\mathrm{P}^m)^{\times s})$

In this brief closing section we summarize our knowledge of examples where Eq. 1.1 is either an equality, or holds within one from being so. On the other hand, we are not aware of any case where Eq. 1.1 actually fails to be an equality.

Since $\mathrm{TC}_s(\mathbb{R}\mathrm{P}^1)=s-1$ ([2, Corollary 3.12]), Equations 4.7 and 4.8 force Eq. 1.1 to be an equality for $m=1$. In slightly more general terms, and as indicated in Example 4.1, equality in Eq. 1.1 holds for $m\in\{1,3,7\}$. It would be interesting to give an explicit construction of the corresponding (forced) $G_s$-maps $\phi_s\colon(S^m)^{\times s}\to J_{s-1}(G_s)$ satisfying Eq. 4.4. For instance, when $s = 2$ and $m = 1$, so that $J_{s-1}(G_s) = S^1$, the required map $\phi_2$ can be defined by multiplication of complex numbers.

In the previous section we have discussed how Theorem 4.8 provides instances with equality in Eq. 1.1 when $m$ is even. We now remark that the same arguments show that, in any case, Eq. 1.1 fails from being an equality by at most a unit provided $m \equiv 1 \bmod 4$ and $s \geq \frac{m+1}{2}$ (as in the case of $m$ even, the restriction imposed by the last inequality can usually be relax substantially).

# Conclusions

Our main contributions in this thesis lie in two branches of topological complexity. The first one is on the calculation of $\mathrm{TC}_s(\mathbb{F}(1^k, m))$, where we found nice bounds for case $s = 2$, and we determine explicitly the value of $\mathrm{TC}_s$ for some suitable cases for $k$ and $m$ (Theorems 3.8 and 3.11, Chapter 3). For those results we developed a good presentation of the cohomology $H^*(\mathbb{F}(1^k, m); \mathbb{Z}_2)$ as a polynomial ring with relations, which are both simple and easily implementable on a computer. Additionally, we saw that when the value of $s$ increases, a stabilization phenomena holds, at least in case of flag manifolds, so it would be interesting to find more examples for that stabilization. At this point, we only use the cohomology mod -2 in our calculations, then it may be possible to use other theories of generalized cohomology (others coefficients, K-theory, cobordism, etc.), and try to improve the bounds on $\mathrm{TC}_s(\mathbb{F}(1^k, m))$, where the calculation of $zcl_s'$ can be done in those cohomologies.

On the other hand, we study the equality $\mathrm{TC}_s(\mathbb{F}(1^k, m)) = Imm(\mathbb{R}\mathrm{P}^m)$ when $s = 2$ and $k = 1$ in order to give a (partial) generalization. The classical result $(s = 2, k = 1)$ is due to Farber-Tabachnikov-Yuzvinsky, and it gives us other point of view on the problem of immersion of projective spaces. Here, we found some evidence to conjecture that $\mathrm{TC}_s(F(1, m))$ equals $secat(\pi_s)$, where $\pi_s$ generalizes $\pi : S^m \times_{\mathbb{Z}_2} S^m \to \mathbb{R}\mathrm{P}^m \times \mathbb{R}\mathrm{P}^m$ (of course, the case $s = 2$ is worked out by FTY). Our first clue in this direction comes from proving that it is possible to construct as many as $secat(\pi_s)+1$, $s$-local rules provided there is a $G_s$-equivariant map $\phi_s$, satisfying Eq. 4.4 together with an additional geometric condition described in 4.4. Other clue is Inequality 4.8, which says that in many cases it can be proved that $G(m, s) = sm - \mathrm{zcl}(\mathbb{R}\mathrm{P}^m) = 0$, for several combinations of parameters, what makes the conjecture true. Recently Don Davis gave an explicit expression for $G(m, s)$, by using explicit calculations based in 2-local cobordism. However it is still an open problem to find $G(m, s)$, which could be accomplished by using others theories of generalized cohomology as the case $\mathrm{TC}_s(\mathbb{F}(1^k, m))$. In both cases there are elements to be developed at greater depth.

Finally, it is worth mentioning another branch of research that can be explored, namely the application of results on new concepts in topological complexity such as equivariant versions of TC.

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