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CONSUMPTION AND PORTFOLIO DECISIONS IN AN ECONOMY WITH HETEROGENEOUS PREFERENCES

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PRESENTA:

ALFREDO OMAR PALAFOX ROCA

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CONSUMPTION AND PORTFOLIO DECISIONS IN AN ECONOMY WITH HETEROGENOUS PREFERENCES

Presentada por la alumna:

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En México D.F., siendo las **10:00** horas el día miércoles 7 del mes **mayo del año 2014**, el (la) que suscribe **Alfredo Omar Palafox Roca** alumno (a) del Programa de **Doctorado en Ciencias Económicas** con número de registro **B101151**, adscrito a la **SEPI ESE-IPN**, manifiesta que es autor (a) intelectual del presente trabajo de Tesis bajo la dirección del **Dr. Francisco Venegas Martínez** y cede los derechos del trabajo intitulado **CONSUMPTION AND PORTFOLIO DECISIONS IN AN ECONOMY WITH HETEROGENOUS PREFERENCES,** al Instituto Politécnico Nacional para su difusión, con fines académicos y de investigación.

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GLOSSARY

Consumption set

Formally, the consumption set is a subset of the commodity space $\mathbb{R}^L$, denoted by $X \subset \mathbb{R}^L$, whose elements are the consumption bundles that the individual can conceivably consume given the physical constraints imposed by his environment.

Decision rule

It is a specification of a procedure, which can be used to identify the best alternative in a any problem. For instance, return maximin rule establishes that the best alternative is one whose worst possible consequence is better than the worst possible consequence arising from any other alternative.

Discount

It is the value of each payment brought to present value trough an interest rate (mostly, a risk free rate).

Distribution function

The distribution function of a random variable $X \subset \Omega$ is the function $F_X(x) = P\{X \leq x\}$ defined for every $x \in \Omega$.

Expectation of a random variable

A random variable $X : \Omega \rightarrow \mathbb{R}$ is said integrable if $\int_\Omega |X| \, dP < \infty$. Then $\mathrm{E}(X) = \int_\Omega X \, dP$ exists and it is called the expectation of *X*.

Expected utility maximization problem

It is the problem that faces the consumer (or a central planner) to choose the optimal consumption path.

Heterogeneous preferences

Given a population, individuals are different among them by their preferences in tastes, risks, endowments, and so on.

Homogeneous preferences

Given a population, all individuals are exactly the same.

Independence of a random variable

Generally speaking, a family, finite or infinite, of random variables is said independent if any finite number of random variables of this family is independent.

Optimal control theory

This theory is complementary to the dynamic optimization problems, integrating variational calculus theory and the optimality principle to the Bellman equation.

Optimal path

When solving the expected utility maximization problem the solution, in terms of the parameters of the utility, usually consumption, is called optimal path.

Present value

This concept is used to referring for a careful deliberation preceding a decision making. Specifically, it concerns qualitative aspects of this decision.

Random variable

If $X \subset \Omega$ and $(\Omega, \mathcal{F}, P)$ is a probability space, then X is a random variable.

Resource allocation

It is the process of dividing available limited resources between different flows of the economy, such as consumption, capital, etc.

Risk

This term means that further consequences of a given decision are unpredictable.

Risk aversion

It is a particular attitude of the consumer when facing uncertainty. He adopts this behavior to reduce the effects of that uncertainty.

Risk free rate

The rate without uncertainty is called a risk free rate.

Subjective discount rate

It is a measure of the anxiety for present and future consumption.

Utility function

A utility function $u : X \to \mathbb{R}$ represents the preference relation $\succsim$ if, for every $x, y \in X$,

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

Welfare function

It is the value of the objective function when substituting the variables involved in the maximization problem.

RESUMEN

Esta tesis considera varios modelos donde una economía es poblada por individuos heterogéneos. La heterogeneidad se origina por dos variables aleatorias: la tasa subjetiva de descuenta y la tasa de aversión al riesgo. Esto es, los consumidores difieren en su nivel de ansiedad por el consumo presente y por su conducta frente al riesgo. La función de utilidad en todos los modelos es del tipo exponencial negativo. Se estudian los modelos sin variables aleatorias, modelos con una sola de estas variables aleatorias y modelos con distribución conjunta. La finalidad de esta investigación es proveer las diferentes trayectorias del consumo óptimo para cada modelo propuesto mediante una fórmula cerrada para dos planteamientos diferentes respecto al tiempo: horizonte a tiempo infinito y finito. Finalmente, se realizan algunos experimentos de estática comparativa.

ABSTRACT

This thesis considers several models where an economy is populated by heterogeneous individuals. Heterogeneity is caused by two random variables: the subjective discount rate and the risk aversion rate. That is, consumers differ in their level of anxiety for present consumption and their behavior towards risk. The utility index is of the negative exponential type. Models without random variables, models with one of these random variables and models with join distribution are studied. The purpose of this research is to provide optimal consumptions paths for each model proposed using a closed formula for two different approaches over time: infinite and finite time horizon. Finally, some comparative statics experiments are carried out.

INTRODUCTION

The subjective discount rate (or discount subjective factor) is a concept used frequently in the literature regarding the utility maximization problem of a rational agent to discount the satisfaction to bring to present value. This concept closely related to intertemporal choice has its origins in economists like John Rae (1834) by introducing psychological factors to explain the differences between the wealth of nations, see, eg, Frederick et al. (2002). Of course, the concept in its early years did not have a mathematical representation as it is known today through an exponential factor (in the continuous time case) with a rate that discounts all the time. Also, the same Frank P. Ramsey (1928) does not use a discount factor exponential or otherwise functional approach for the intertemporal decision problem.

However, Adam Smith (1764) assumes in his research that the wealth of nations is due to the amount of work assigned to the production of capital. Rae noted in that explanation some drawbacks; in particular, Smith does not specify what the determinants for this assignment are. Rae's response, like that of von Böhm Bawerk (1889) of the Austrian school, proposed to be psychological factors on the desire for accumulation that determines the allowance, see Epstein and Hynes (1983). This implies that the designation of resources depends to some extent on subjective factors.

Samuelson (1937) to take up the idea of the subjectivity in defining marginal utility proposes a model in which an individual maximizes the sum of all discounted future profits with respect to time, along with other assumptions. Moreover, Samuelson provides mathematical relationships that reflect the behavior of agents and introduces a discount parameter. He identifies a number of deficiencies in the model inherently has since discounted utility as would a certain amount of money, in spite of that, economists of the time accepted the proposal favorably for its ease of use and the advantages for mathematical calculations. Some empirical research that attempt to measure the level of impatience (or anxiety) of consumers for current consumption are in Booij and Van Praag, 2003 and Epstein and Hynes, 1983.

This thesis develops a model with an economy populated by heterogeneous agents. Heterogeneity refers to individuals with different preferences (or tastes) in two respects. First we assume that the parameter that represents the subjective discount rate is exponentially distributed. Second parameter assumes the depth, i.e., the parameter in the utility function also has an exponential probability distribution, which is assumed negative exponential type. Therefore, consumers who populate the economy differ in their level of anxiety for present consumption and its utility function[1].

In other words, consumers have heterogeneous preferences. The heterogeneity has been studied from various angles, primarily in valuation of financial assets. Given the implausibility assuming that all asset returns are multivariate normal and a stochastic distribution of wealth among consumers Constantinides (1982) solves this problem by considering heterogeneous consumers, such that the proposed solution coincides in equilibrium with the central planner solution. Constantinides and Duffie (1996) reach equilibrium in an economy with heterogeneity in the form of uninsurable labor income shocks, persistent and heteroskedastic.

Chan and Kogan (2001) characterize the competitive equilibrium in an exchange economy in which individual agents have to update their preferences regarding the preferences of others, and differ only with respect to the curvature of their utility functions. For Judd et al (2003) heterogeneity is represented by a matrix in terms of the proportion of the wealth of individuals in different assets. In Basak's model (2003) in addition to valuing the fundamental risk, the non fundamental risk is valued by the agent too, with a market price that represents a risk-tolerance weighted average of his extraneous disagreement with all remaining agents; in this case the heterogeneity is modeled through arbitrary utility functions.

Luttmer and Marioti (2003) provide a convenient analytical approximation of continuous time and show how subjective rates of time preference affect risk-free rates but not instantaneous risk-return trade-offs. To Jouini and Napp (2007) the objective is to analyze the impact of heterogeneous beliefs in a complete competitive market economy, but standard otherwise. The construction of a consensus probability belief, and the consensus of consumers, is shown to be valid modulo an aggregation bias, which takes the form of a discount factor. This discount factor makes the heterogeneous beliefs setting fundamentally

[1] Another paper dealing with different individuals, for example, age, disease, probability of survival or other factors are found in Li-Wei *et al* (2009).

different from the scenario of homogeneous beliefs, and is consistent with the interpretation of the heterogeneity of beliefs as a source of risk.

Follmer *et al* (2005) presented a schema with switches in beliefs when individuals move from one period to another, heterogeneity is represented by a probability. On the other hand, the Lucas model of asset prices in an exchange economy is modified by Li (2007), who presents a new proposal, which is based on the modification of the homogeneous environment. In this model, all investors have logarithmic preferences and different subjective discount rates. This closed-form solution allows us to analyze the dynamics of stock prices and their volatility.

Kuplow (2008) analyzes optimal policies when the preferences for commodities, public goods and externalities are heterogeneous; Shapiro (2008) studies the overvaluation of a risky asset in a framework with heterogeneous agents with non-rational expectations; Chen *et al.* (2008) consider heterogeneity in preferences over a local public good, human capital formation, and residential locations through an overlapping generation model; Fethke and Jagannathan (1996) examine the dynamics of consumption in a setting where imperfectly competitive producers face consumers with various intensities of rational habit persistence; Boswijk *et al.* (2007) estimate a dynamic asset pricing model characterized by heterogeneous bounded rational agents; Andersen (2007) analyzes an intertemporal general equilibrium model with heterogeneous labor markets; and Xiouros and Zapatero (2010) study economies populated with agents with heterogeneous risk aversion.

This research focuses on the decision making process of the average consumer of an economy populated by individuals differing in their preferences. First, a finite horizon is analyzed. Later, the case of an infinite horizon is studied. Specifically, heterogeneity is introduced via a joint distribution function of the subjective discount rate and the risk aversion coefficient; both parameters being driven by the exponential distribution. We also suppose that individuals are endowed with a negative exponential utility function. This functional form is appropriate to be conjoined with the exponential density so that the discounted total utility of the average consumer can be analytically treated. One distinguishing feature of this research is that closed-form solutions for the optimal paths of consumption and capital, of the average consumer, are obtained in both frameworks. Furthermore, a closed form solution for the economic welfare of the average consumer is derived for each case. Finally, several analytical and graphical experiments of comparative statics are accomplished.

CHAPTER 1. REVIEW OF PROBABILITY CONCEPTS

Basic and fundamental concepts of probability theory are presented in this first chapter. To summarize topics are reviewed as probability, random variable, among others. These concepts are some of the foundations of the models developed through the thesis. Several references as Hernández-Lerma (2009), Shreve (2004), Chung (2001), Papoulis (1991), Billingsley (1986) and Boes *et al* (1974) were consulted to structure this review.

1.1 Events and probability

Let Ω be a nonempty set, and let $\mathcal{F}$ be a collection of subsets of Ω. Then, it is said that $\mathcal{F}$ is a σ-algebra provided that:

(a) $\Omega \in \mathcal{F}$,

(b) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$,

(c) If $\{A_1, A_2, \ldots\}$ is a sequence of sets in $\mathcal{F}$, then $\cup A_n \in \mathcal{F}$.

Condition (c) only applies to countable sequences, specifically to countable family of events, not to an arbitrary family of events $\mathcal{F}$. Usually, the pair $(\Omega, \mathcal{F})$ is named as a measurable space because it is possible to define a measure over such set. If A is a set in $\mathcal{F}$, then it is said that A is $\mathcal{F}$-measurable. In probabilistic terms, Ω is the sample space, and every $A \in \mathcal{F}$ is called an event. The complement of Ω is named the empty set, $\varnothing$; condition (b) guarantees that $\varnothing \in \mathcal{F}$.

Let $\mathcal{F}$ be a σ-algebra in Ω. A probability measure is a function $P : \mathcal{F} \to [0,1]$ such that:

1) $P(\Omega) = 1$;

2) $P(A) \geq 0$, $A \in \mathcal{F}$;

3) If $(A_n)_{n\in\mathbb{N}}$ is a sequence of disjoint events (i.e. $A_i \cap A_j = \varnothing$ for $i \neq j$, $i,j \in \mathbb{N}$) then it holds that $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$.

Incorporating P to a measurable space, as defined earlier, gives as a result a probability space, $(\Omega, \mathcal{F}, P)$.

1.2 Random variables

Let $\mathcal{F}$ be a σ-algebra in Ω, a function $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-measurable if $\{X \in \mathrm{B}\} \in \mathcal{F}$ for every Borel set $\mathrm{B} \in \mathcal{B}(\mathbb{R})$, such that $\mathcal{B}(\mathbb{R})$ represents the Borel σ-algebra, *i.e.*, the σ-algebra of the open sets of $\mathbb{R}$. If $(\Omega, \mathcal{F}, P)$ is a probability space, then X is a random variable. A σ-algebra generated by a random variable X, $\sigma(X)$, consists of all the sets of the form $\{X \in \mathrm{B}\}$, where B is a Borel set in $\mathbb{R}$. Every random variable, $X : \Omega \to \mathbb{R}$, leads to a probability measure, $P_X(\mathrm{B}) = P\{X \in \mathrm{B}\}$, in $\mathbb{R}$ defined over the σ-algebra of the Borel sets $\mathrm{B} \in \mathcal{B}(\mathbb{R})$. P_X is called the distribution of X. The function $\mathrm{F}_X : \mathbb{R} \to [0,1]$ defined for $\mathrm{F}_X(x) = P\{X \leq x\}$ is called the distribution function of X.

It is standard in probability theory to call a function a $f : \mathbb{R} \to \mathbb{R}$ as a Borel function if the inverse image $f^{-1}(\mathrm{B})$ of any Borel set in $\mathbb{R}$ is also a Borel set. If there exists a Borel function $f_X : \mathbb{R} \to \mathbb{R}$ such that for any Borel set $\mathrm{B} \subset \mathbb{R}$,

$$P\{X \in \mathrm{B}\} = \int_{\mathrm{B}} f_X(x)dx,$$

then X is a random variable with absolutely continuous distribution and f_X is called the density of X. If there exists a sequence, finite or infinite, of distinct real numbers $x_1, x_2, \dots$ such that for any Borel set $B \subset \mathbb{R}$,

$$P\{X \in B\} = \sum_{x_i \in B} P\{X = x_i\},$$

then X has discrete distribution with values $x_1, x_2, \dots$ and mass $P\{X = x_i\}$ in x_i.

The joint distribution of several random variables $X_1, \dots, X_n$ is a probability measure $P_{X_1 \cdots X_n}$ in $\mathbb{R}^n$ such that $P_{X_1 \cdots X_n}(B) = P\{(X_1 \cdots X_n) \in B\}$ for any Borel set B in $\mathbb{R}^n$. If there exists a function $f_{X_1 \cdots X_n} : \mathbb{R}^n \to \mathbb{R}$ such that

$$P\{(X_1 \cdots X_n) \in B\} = \int_B f_{X_1 \cdots X_n}(X_1 \cdots X_n) dx_1 \cdots dx_n$$

for every Borel set in $\mathbb{R}^n$, then $f_{X_1 \cdots X_n}$ is called the joint density of $X_1, \dots, X_n$.

A random variable $X : \Omega \to \mathbb{R}$ is said integrable if $\int_\Omega |X| \, dP < \infty$. Then $\mathrm{E}(X) = \int_\Omega X \, dP$ exists and it is called the expectation of X. In real analysis, a family of integrable random variables is denoted by L^1 or, when ambiguity is possible, by L^1 $(\Omega, \mathcal{F}, P)$. A random variable $X : \Omega \to \mathbb{R}$ is called square integrable if $\int_\Omega |X|^2 d\mathrm{P} < \infty$. The variance of X can be defined as follows: $\mathrm{Var}(X) = \int_\Omega (X - \mathrm{E}(X))^2 d\mathrm{P}$. The family of square integrable random variables is denoted by L^2 or, when ambiguity is possible, by L^2 $(\Omega, \mathcal{F}, P)$.

1.3 Conditional probability and independence

For any events $A, B \in \mathcal{F}$ such that $P(B) \neq 0$ the conditional probability of A given B is defined by $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$. Two events $A, B \in \mathcal{F}$ are named independent if

$P(A \cap B) = P(A) \cdot P(B)$. In general, it is said that n events $A_{i1}, A_{i2}, \cdots, A_{ik} \in \mathcal{F}$ are independents if

$$P(A_{i1} \cap A_{i2} \cap \cdots \cap A_{ik}) = P(A_{i1}) P(A_{i2}) \cdots P(A_{ik})$$

for any indexes $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. Two random variables X y Y are independents if for every Borel sets A, $B \in \mathcal{B}(\mathbb{R})$ the events $\{X \in A\}$ and $\{Y \in B\}$ are independents. In a similar form, n random variables $X_1, X_2, \ldots, X_n$ are independents if for every Borel set B_1, B_2, …, $B_n \in \mathcal{B}(\mathbb{R})$ the events $\{X_1 \in B_1\}$, … , $\{X_n \in B_n\}$are independents.

Generally speaking, a family, finite or infinite, of random variables is said independent if any finite number of random variables of this family is independent. If two integrable random variables $X_1, Y : \Omega \to \mathbb{R}$ are independents, then they are not correlated, *i.e.*, $\mathrm{E}(X\,Y) = \mathrm{E}(X)\,\mathrm{E}(Y)$, provided that XY is also integrable. If the random variables $X_1, X_2, \ldots, X_n : \Omega \to \mathbb{R}$ are independent, then $\mathrm{E}(X_1 X_2 \cdots X_n) = \mathrm{E}(X_1)\,\mathrm{E}(X_2) \cdots E(X_n)$, whenever the product of the n random variables is integrable. Furthermore, two σ-algebras $\mathcal{G}$ and $\mathcal{H}$ contained in $\mathcal{F}$ are independents if any two events $A \in \mathcal{G}$ and $B \in \mathcal{H}$ are independents. Similarly, any finite number of σ-algebras $\mathcal{G}_1$, … , $\mathcal{G}_n$ contained in $\mathcal{F}$ are independents if any n events $A_1 \in \mathcal{G}_1$, … , $A_n \in \mathcal{G}_n$ are independents. Generally, a family, finite or infinite, of σ-algebras is said independent if every finite number of them is independent.

1.4 Distribution and density function of a continuous random variable

Let X be a subset of $\Omega = \mathbb{R}$, then the elements of Ω that are contained in the event $\{X \leq x\}$ change as the number x takes various values. The probability $P\{X \leq x\}$ is, therefore, a number that depends of x. This number is denoted by $F_X(x)$ and is called the (cumulative) distribution of the random variable X. So, the definition of a distribution function of the random variable X is the function $F_X(x) = P\{X \le x\}$ defined for every $x \in \Omega$. Usually, if there is no fear of ambiguity the subscript is omitted.

Proposition 1.4.1 The distribution function has the following properties:

1. i) $F(\infty) = 1$, and ii) $F(-\infty) = 0$.

Proof

i) $F(\infty) = P\{X \le \infty\} = P\{\Omega\} = 1$.

ii) $F(-\infty) = P\{X \le -\infty\} = P\{\varnothing\} = 0$.

2. It is a non decreasing function of *x*: if $x_1 < x_2$, then $F(x_1) \le F(x_2)$.

Proof

It is obvious that $\{X \le x_1\} \subseteq \{X \le x_2\}$. Next, since by definition $F(x) = P\{X \le x\}$, it follows that $F(x_2) = P\{X \le x_2\} = P\{X \le x_1\} + P\left\{X \le x_2 \cap (X \le x_1)^c\right\} \ge P\{X \le x_1\} = F(x_1)$.

3. If $F(x_0) = 0$, then $F(x) = 0$ for every $x \le x_0$.

Proof

Apply property 2.

4. $P\{X > x\} = 1 - F(x)$.

Proof

Since $\Omega = \{X \le x\} \cup \{X > x\}$. then $1 = P\{\Omega\} = P\{X \le x\} + P\{X > x\}$. Considering the definition of a distribution function and rearranging the sum, $P\{X > x\} = 1 - F(x)$.

5. $P\{x_1 < X \le x_2\} = F(x_2) - F(x_1)$.

Proof

It is easy to verify that $\{x_1 < X \le x_2\}$ and $\{X \le x_1\}$ are disjoint events. Additionally, property 2 is applied in order to get an expression that relates the sets involved, like this $P\{X \le x_2\} = P\{X \le x_1\} + P\{x_1 < X \le x_2\}$. Finally, following the definition of a distribution function, $P\{x_1 < X \le x_2\} = F(x_2) - F(x_1)$.

6. The function $F_X(x)$ is continuous from the right

Proof

As $\{x < X \leq x + h\} \downarrow \emptyset$ when $h \downarrow 0$, $F(x+h) - F(x) = P\{x < X \leq x+h\} \downarrow 0$.

7. $P\{X = x\} = F(x) - F(x^-)$, with $x^- = \lim_{\varepsilon \to 0} x - \varepsilon$.

Proof

Set $x_2 = x$ and $x_1 = x - \varepsilon$, then from property 5 it follows that $P\{x - \varepsilon < X \leq x\} = F(x) - F(x - \varepsilon)$, and with $\varepsilon \to 0$ property 7 holds.

8. $P\{x_1 \leq X \leq x_2\} = F(x_2) - F(x_1^-)$.

Proof

It follows from property 5 and property 7, because $\{x_1 < X \leq x_2\}$ and $\{X = x_1\}$ are disjoint events.

A random variable *X* is continuous if its distribution function $F(x)$ is continuous. In this case, $F(x^-) = F(x)$; hence, $P\{X = x\} = 0$ for every *x*. The derivative of a continuous distribution function, $F(x)$, is called the density function

$$f(x) = \frac{dF(x)}{dx}.$$

From the monotonicity of $F(x)$ it follows that $f(x) > 0$. Integrating the last derivative, it is obtained

$$F(x) = \int_{-\infty}^{x} f(s)\,ds.$$

By property 1, the above yields

$$\int_{-\infty}^{\infty} f(s)\,ds = 1.$$

Consequently, from the definition of a distribution function of a continuous random variable it follows that

$$F(x_2) - F(x_1) = \int_{x_1}^{x_2} f(s)\,ds.$$

Hence,

$$P\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f(s)\,ds.$$

The present thesis proposes an environment with heterogeneity, which is produced by continuous random variables, all of them with exponential distribution function. If a random variable *X* has a density given by

$$f_X(x) = \lambda e^{-\lambda x},$$

where $\lambda > 0$, then *X* is defined to have an exponential distribution. In order to prove that it is a distribution function it is necessary to integrate $f_X(x)$ and verify that integrates up to 1,

$$\int_0^\infty \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^\infty = 1.$$

Next, the expectation of this random is calculated as follows:

$$E[X] = \int_0^\infty \lambda x e^{-\lambda x}\,dx = -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx = -\frac{1}{\lambda}e^{-\lambda x}\Big|_0^\infty = \frac{1}{\lambda}.$$

The second moment,

$$E\left[X^2\right] = \int_0^\infty \lambda x^2 e^{-\lambda x}\,dx = -x^2 e^{-\lambda x}\Big|_0^\infty + \int_0^\infty 2xe^{-\lambda x}\,dx = \frac{2}{\lambda}\int_0^\infty \lambda x e^{-\lambda x}\,dx = -\frac{2}{\lambda^2}e^{-\lambda x}\Big|_0^\infty = \frac{2}{\lambda^2}.$$

Finally, the variance

$$Var[X] = E\left[X^2\right] - \left(E[X]\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Usually, a compact notation to say that a random variable *X* is distributed exponential is as follows: $X \sim \exp(\lambda)$.

1.5 Fubini's theorem

In chapters 3 to 5 the different models have something in common: all of them need Fubini's theorem to be solved, chapters 4 and 5 use it twice. This section is about what the theorem establishes. However, the proof of the theorem is not developed since the main topic of the models proposed in this thesis only requires it as a tool, which gives facility to exchange the order of integration. Thus, it is possible to face a problem with simpler integrals, or at least not so difficult to solve taking into account that further maximization problems deal with two or three measures.

Fubini's Theorem. Let $(\Omega_i, \mathcal{F}_i, \mu_i)$, i = 1, 2 be a σ-finite measure spaces and let

$f \in L^1(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$. Then there exist sets $B_i \in \mathcal{F}_i$, i = 1, 2 such that

i) $\mu_i(\Omega_i \setminus B_i) = 0$ for i = 1, 2.

ii) for $\omega_1 \in B_1$, $f(\omega_1, \cdot) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2)$,

iii) the function

$$g_1(\omega_1) \equiv \begin{cases} \int_{\Omega_2} f(\omega_1, \omega_2)\mu_2(d\omega_2) & \text{for } \omega_1 \in B_1 \\ 0 & \text{for } \omega_1 \in B_1^c \end{cases}$$

is $\mathcal{F}_1$-measurable and

$$\int_{\Omega_1} g_1 d\mu_1 = \int_{\Omega_1 \times \Omega_2} f d(\mu_1 \times \mu_2),$$

iv) for $\omega_2 \in B_2$, $f(\cdot, \omega_1) \in L^1(\Omega_1, \mathcal{F}_1, \mu_1)$,

v) the function

$$g_2(\omega_2) \equiv \begin{cases} \int_{\Omega_1} f(\omega_1, \omega_2)\mu_1(d\omega_1) & \text{for } \omega_2 \in B_2 \\ 0 & \text{for } \omega_2 \in B_2^c \end{cases}$$

is $\mathcal{F}_1$-measurable and

$$\int_{\Omega_2} g_2 d\mu_2 = \int_{\Omega_1 \times \Omega_2} f d(\mu_1 \times \mu_2).$$

For a complete proof of this theorem see Krishna and Soumendra (2008).

CHAPTER 2. ECONOMIC RATIONALITY

This chapter is based on Microeconomic Theory (Green, 1995) and Advanced Microeconomic Theory (Jehle and Reny, 2001). This review begins by showing the study of the theory of decision making of an individual. The starting point for any decision problem of an individual is a set of possible alternatives (mutually exclusive), from which he must choose. Denote this set of alternatives as *X*.

2.1 Preferences

There are two different approaches to model choice behavior of an individual. The first is called the preference relation and the second approach, based on the axiom of revealed preference, is called choice rule. As the next chapter examines the issue of utility theory is appropriate to follow the approach of preference relations, since this theory is based on these relationships. The objectives of the decision maker are summarized in a preference relation, which we denote by $\succsim$. Technically, $\succsim$ it is a binary relation on the set of alternatives *X*, allowing the comparison of alternative pairs $x, y \in X$. The expression $x \succsim y$ is read as "x is at least as preferred as y". From $\succsim$, we can derive two important relations on *X*.

i) The strict preference relation, $\succ$, defined by

$$x \succ y \Leftrightarrow x \succsim y \text{ but not } y \succsim x$$

and it is read as "*x* is strictly preferred to *y*".

ii) The indifference relation, $\sim$, defined by

$$x \sim y \Leftrightarrow x \succsim y \text{ and } y \succsim x$$

and it is read as "*x* is indifferent to *y*".

In most of microeconomic theory, individual preferences are assumed to be rational. The assumption of rationality is built on two basic assumptions about the preference relation, $\succsim$: completeness and transitivity.

Definition 2.1.1: The preference relation $\succsim$ is rational if it possesses the following two properties:

i) Completeness: for all $x, y \in X$, we have that $x \succsim y$ or $y \succsim x$ (or both).

ii) Transitivity: for all $x, y, z \in X$, if $x \succsim y$ y $y \succsim z$, then $x \succsim z$.

The assumption that the preference relation is complete says that the individual has a well-defined preference between any two possible alternatives. The completeness axiom establishes that the decision makers make only meditated choices. The transitivity is also a strong assumption, and it goes to the heart of the concept of rationality. Transitivity implies that it is impossible to face the decision maker with a sequence of pairwise choices in which her preference appear to cycle. The assumption that the preference relation $\succsim$ is complete and transitive has implications for the strict preference and indifference relations.

Proposition 2.1.1: If $\succsim$ is rational then:

i) $\succ$ is both irreflexive ($x \succ x$ never holds) and transitive (if $x \succ y$ and $y \succ z$, then $x \succ z$).

ii) $\sim$ is reflexive ($x \sim x$ for all x), transitive (if $x \sim y$ and $y \sim z$, then $x \sim z$) and symmetric (if $x \sim y$, then $y \sim x$).

iii) if $x \succ y \succsim z$, then $x \succ z$.

Proof

i) From the definition of $\succ$ it follows that $x \succsim x \Leftrightarrow x \succsim x$ but not $x \succsim x$, which it is a contradiction, then $x \succsim x$ does not hold. To verify that transitivity does hold in the strict preference relation $\succ$ take x, y, $z \in X$, with $x \succsim y$ y $y \succsim z$, then $x \succsim z$. It follows from the definition of $\succ$ that if $x \succsim y$, then it is not possible that $y \succsim x$, analogously with $y \succsim z$, this implies that $z \succsim y$ does not hold; finally, the last relation follows by transitivity: $x \succ y$ y $y \succ z$, therefore, $x \succ z$.

ii) By definition of $\sim$ and since $\succsim$ is rational, it follows that $x \succsim x$ and $x \succsim x$, then $x \sim x$. Transitivity results to apply, once again, the definition of $\sim$. Suppose that x, y, $z \in X$, such that $x \succsim$ and y $y \succsim z$, then $x \succsim z$. On the other hand, $x \succsim y$ and $y \succsim x$ imply that $x \sim y$,

analogously $y \sim z$, so $x \sim z$. Symmetry is a consequence of rearranging the relations, *i.e.*, $y \succsim x$ and $x \succsim y$, but this is the definition of the indifference relation, so $y \sim x$.
iii) If $x \succ y \succsim z$, then two things can happen. First, that $y \succ z$, and by transitivity, $x \succ z$. Second, $y \sim z$, whereby it follows that $x \succ z$.

The last proposition is important, since the rationality of $\succsim$ implies that both relations, $\succ$ and $\sim$, are transitive. It can be seen that transitivity can be extended by combining the relations, as in case iii) of the previous proposition. As well as this combination there are other combinations, which rise from the definition of preference relation, strict preference relation and indifference relation.
In addition to the assumptions mentioned earlier there are two more about the consumer preferences: desirability and convexity.
i) Desirability. Sometimes it is reasonable to assume that larger quantities of commodities are preferred to small quantities. This issue is captured by the monotonicity assumption. Suppose that consumption of large quantities of goods is feasible, this is, if $x \in X$ and $y \geq x$, then $y \in X$.

Definition 2.1.2: The preference relation $\succsim$ on X is monotone if $x \in X$ and $y \gg x$ implies that $y \succ x$. It is strongly monotone if $y \geq x$ y $y \neq x$ implies that $y \succ x$. This assumption is satisfied as long as the commodities are goods, not bads. Note that if the preference is monotone, then indifference respect to an increase in the quantity of certain goods is likely to occur, but not in every good. In contrast, strict monotonicity says that if *y* is larger than *x* for a given commodity and it is not less for any other, then *y* is strictly preferred to *x*. However, most of the theory uses a weaker assumption of desirability known as locally nonsatiation.

Definition 2.1.3: The preference relation $\succsim$ on X is locally nonsatiated if for every $x \in X$ and every $\varepsilon > 0$, there exists $y \in X$ such that $\|y-x\| \leq \varepsilon$ and $y \succ x$.

ii) Convexity. An important second assumption deals with the convexity of the preference relation. That is, the exchange that the consumer is willing to make between the different goods.

Definition 2.1.4: The preference relation on X is convex if for every $x \in X$, the upper contour $\{y \in X : y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1-\alpha)z \succsim x$ for any $\alpha \in [0,1]$. Convexity is a central and strong hypothesis in economics. Convexity can be seen as the formal expression of a basic propensity of the economic agents for diversification. In fact, under convexity, if x is indifferent to y, then $0.5\,x + 0.5\,y$, the half-half mixture of x and y, it cannot be worse than both, x or y. Economic theory would be in serious difficulties if the propensity of this postulate for diversification would not have a meaningful descriptive content. In spite of this, there is no doubt that an individual can easily figure out choice cases where the postulated is violated.

Definition 2.1.5: A preference relation on X is strictly convex if for every x the following happens, $y \succsim x$, $z \succsim x$, and $y \neq z$ implies that $\alpha y + (1-\alpha)z \succ x$ for any $\alpha \in [0,1]$.

2.2 Utility function

In economics, frequently, preference relations are described by means of a utility function. A utility function $u(x)$ assigns a numeric value to each element on X, ordering the elements of X accordingly to individual's preferences.

Definition 2.2.1: A function $u : X \to \mathbb{R}$ is a utility function that represents the preference relation $\succsim$, if for every $x, y \in X$,

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

Note that a utility function is not unique. For any function strictly increasing $f : \mathbb{R} \to \mathbb{R}$, $v(x) = f(u(x))$ is a new utility function that represent the same preferences in $u(\cdot)$, only it matters the order. The utility function properties which are invariant to any strictly increasing transformation are called ordinals. Cardinal properties are those that do not preserve the same order under these transformations. Thus, the preference relations associated with a utility function is an ordinal property. On the other hand, numerical values associated with alternatives on *X*, and therefore the magnitude of any difference in the utility measure among alternatives, are cardinal properties.

Proposition 2.2.1: A preference relation $\succsim$ can be represented by a utility function only if it is rational.

Proof: To prove this proposition it is shown that there exists a utility function that represents $\succsim$, and then $\succsim$ must be complete and transitive.

Because $u(\bullet)$ is a real-valued function defined on *X*, it must be that for any $x, y \in X$, $u(x) \geq u(y)$ or $u(y) \geq u(x)$. But because $u(\bullet)$ is a utility function representing $\succsim$, this implies either that $x \succsim y$ or $y \succsim x$. Hence, $\succsim$ must be complete.

Suppose that $x \succsim y$ and $y \succsim z$. Because $u(\bullet)$ is a function representing $\succsim$, it must be that $u(x) \geq u(y)$ and $u(y) \geq u(z)$. Hence, $u(x) \geq u(z)$. Since $u(\bullet)$ represents $\succsim$, this implies that $x \succsim z$. Thus, this shows that $x \succsim y$ and $y \succsim z$ implies $x \succsim z$ and the transitivity is established.

Definition 2.2.2. The preference relation on X is continuous if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for any *n*, $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, it must be $x \succsim y$. Continuity says that preference cannot exhibit jumps. In other words, the upper contour $\{ y \in X: y \succsim x \}$ and the inferior contour $\{ y \in X: x \succsim y \}$ are both closed.

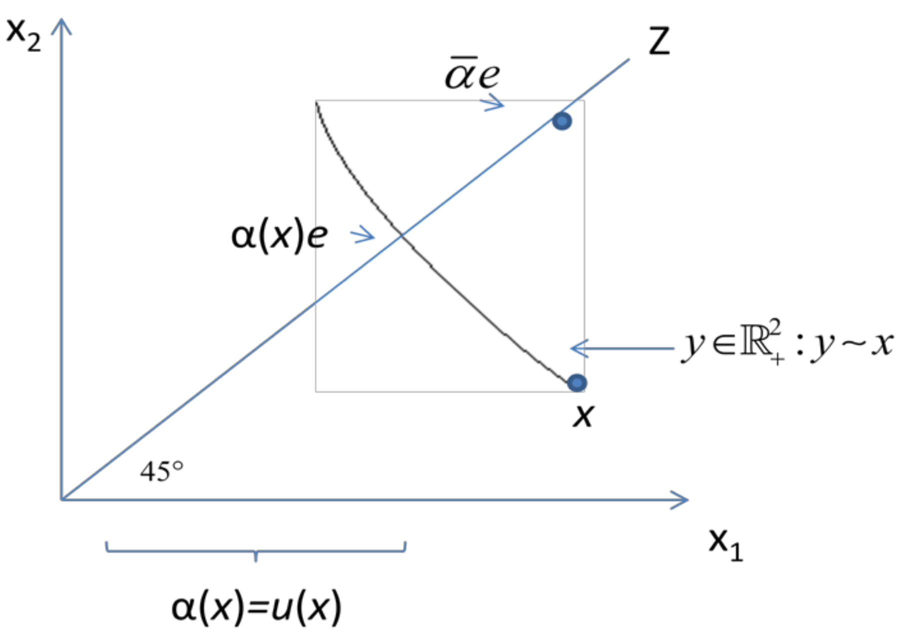
Proposition 2.2.2: Suppose that the rational preference relation $\succsim$ on X is continuous. Then there exist a utility function representing $\succsim$.

Proof. For the case $X = \mathbb{R}_+^L$ and a monotone preference relation, there exists a relatively simple and intuitive proof represented by means of the next graph.

Denote the diagonal ray in $\mathbb{R}_+^L$ by Z. It will be convenient to let e designate the L-vector whose elements are all equal to 1. Then $\alpha e \in Z$ for all nonnegative scalars $\alpha > 0$. Note that for every $x \in \mathbb{R}_+^L$, monotonicity implies that $x \succsim 0$. Also note that for any $\bar{\alpha}$ such that $\bar{\alpha} e \gg x$, it must be that $\bar{\alpha} e \succsim x$. Monotonicity and continuity can then be shown to imply that there is a unique value $\alpha(x) \in [0, \bar{\alpha}]$ such that $\alpha(x) e \sim x$.

Take $\alpha(x)$ as our utility function, that is, assign a value of utility $u(x) = \alpha(x)$ to every x. It is necessary to verify two properties of this function: first, that represents a preference relation $\succsim$; second, that it is a continuous function.

By construction $\alpha(x)$ represents preferences. Finally, suppose that $\alpha(x) \geq \alpha(y)$. By monotonicity, this implies that $\alpha(x) e \succsim \alpha(y) e$.



Graph 2.1. Construction of a utility function. Source: Green (1995).

Since $x \sim \alpha(x) e$ and $y \sim \alpha(y) e$, then it follows that $x \succsim y$. On the other hand, assume that $x \succsim y$. Then $\alpha(x) e \sim x \succsim y \sim \alpha(y) e$; and by monotonicity, it must be that $\alpha(x) \geq \alpha(y)$. Hence, $\alpha(x) \geq \alpha(y) \Leftrightarrow x \succsim y$.

2.3 Competitive budget

In addition to physical constraints embodied in the consumption set, the consumer faces a major economic constraint: his choice is limited to those commodity bundles he can afford. To formalize this restriction, two assumptions are introduced. First, it is assumed that the *L* commodities are all traded in the market at a given monetary unit prices that are publicly quoted. Formally, these prices are represented by a vector of prices.

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L,$$

which gives the monetary unit cost for a unit of each of the *L* commodities. Observe there is nothing that logically requires prices to be positive. A negative price simply means that a buyer is actually paid to consume the commodity (which is not illogical for commodities that are bads, such as pollution). Nevertheless, for simplicity, it is assumed that $p \gg 0;$ that is, $p_l > 0$ for every *l*.

Second, it is assumed that these prices are beyond the influence of the consumer. There is the so called price-taking assumption. Loosely speaking, this assumption is likely to be valid when the consumer's for any commodity represents only a small fraction of the total demand for that good.

The affordability of a consumption bundle depends on two things: the market prices $p = (p_1, \ldots, p_L)$ and the consumer's wealth level (in monetary units) *w*. The consumption bundle $x \in \mathbb{R}_+^L$ is affordable if its total cost does not exceed the consumer's wealth level *w*, that is, if

$$p \cdot x = p_1 x_1 + \cdots + p_L x_L \leq w.$$

The economic-affordability constraint, when combined with the requirement that *x* lie in the consumption set $\mathbb{R}_+^L$, implies that the set of feasible consumption bundles consists of the elements of the set $\{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$. This set is known as the Walrasian, or competitive budget set.

Definition 2.3.1: The Walrasian or competitive budget set $B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$ is the set of all feasible consumption bundles for the consumer who faces market prices *p* and has wealth *w*. Then, the consumer´s problem, given prices *p* and wealth *w*, can thus be stated as follows: choose a consumption bundle *x* from $B_{p,w}$.

The Walrasian budget set $B_{p,w}$ is a convex set: that is, if the bundles *x* and x' are both elements of $B_{p,w}$, then the bundle $x'' = \alpha\, x + (1\text{-}\,\alpha)\, x'$ is also an element of $B_{p,w}$. Note that because both *x* and x' are nonnegative, $x'' \in \mathbb{R}_+^L$. Further, every time that $p \cdot x \leq w$ and $p \cdot x' \leq w$, it follows that $p \cdot x'' = \alpha(p \cdot x) + (1-\alpha)(p \cdot x') \leq w.$ Therefore, $\{x'' \in B_{p,w} = x \in \mathbb{R}_+^L : p \cdot x \leq w\}$.

2.4 Utility maximization

Under the previous assumptions the feasible set, total expenditure must not exceed income. Formally, the utility maximization problem is written

$$\underset{x\geq 0}{\text{Maximize}} \quad u(x)$$
$$\text{subject to} \quad p \cdot x \leq w.$$

The solution of this problem consists that the consumer chooses a consumption bundle on the Walrasian budget set, $B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$ to maximize his utility level.

Proposición 2.4.1: If $p \gg 0$ and $u(\cdot)$ is continuous, then the maximization problema has a solution.

Proof

If $p \gg 0$, then the budget set $B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$ is a compact set because it is both bounded [for any *l* = 1, …, *L*, we have $x_l \leq w/p_l$ for all $x \in B_{p,w}$] and closed. The result follows from the fact that a continuous function always has a maximum value on any compact set.

Throughout this thesis all models consider economies in continuous time. The theoretical foundations of each model lie in the definitions and propositions studied in the present chapter. The problem settings have special structures that generate optimal solutions for consumption and capital.

CHAPTER 3. CONVENTIONAL HOMOGENEOUS CONSUMPTION AND RANDOMIZATION OF *ρ* AND *α*.

Chapter 3 presents the homogeneous case, based on a number of assumptions. In order to expose the results of this model it is useful introduce previous background. Firstly, the conventional nonrandomized problem is introduced. Secondly, the analysis focuses on the randomization of the subjective discount rate. Thirdly, the rate of risk aversion is considered a random variable. Furthermore, the first three sections analyze the infinite time horizon, whilst the last three sections deal with the finite time horizon.

3.1 Conventional model: infinite time horizon (ITH)

First, let us recall the conventional model of discounted utility, $u\left(c_t\right)$, with infinite horizon and negative exponential utility function: $-e^{-\alpha c_t}$, $\alpha > 0$. It is assumed that α is a known parameter. The reason to choose this function is because of its concavity property. The discounted utility maximization problem facing the consumer, with a subjective discount rate ρ, it is established in the next form:

$$\text{Maximize} \quad \int_0^\infty u\left(c_t\right)e^{-\rho t}dt$$

$$\text{subject to } k_0 = \int_0^\infty c_t e^{-rt}dt. \qquad (3.1.1)$$

Where k_0 is the initial endowment which in this case is constant and c_t is the consumption at time t; the restriction in (3.1.1) indicates that the endowment and the discounted consumption, with a real interest rate, are the same.

Omitting the boundary conditions, Lagrangian is given by:

$$\mathfrak{L}\left(c_t, \lambda\right) = -e^{-\alpha c_t}e^{-\rho t} + \lambda\left(rk_0 - c_t\right)e^{-rt}$$

where λ is a constant to be determined. Deriving with respect to c_t, it follows that

$$\alpha e^{-\alpha c_t}e^{-\rho t} - \lambda e^{-rt} = 0.$$

The above equation leads to

$$c_t = -\frac{1}{\alpha}\ln\left(\frac{\alpha}{\lambda}\right) + \frac{1}{\alpha}\left(r - \rho\right)t \qquad (3.1.2)$$

The last expression shows only known parameters a priori, but λ, which is the Lagrange's multiplier, and this must be isolated in order to obtain the optimal path through time. To find this value, the right hand side of the restriction in (3.1.1) is replaced by equation (3.1.2) and the integral is solved. Once the integral is solved, λ is isolated and is substituted into (3.1.2). The development of the math is shown in appendix A.

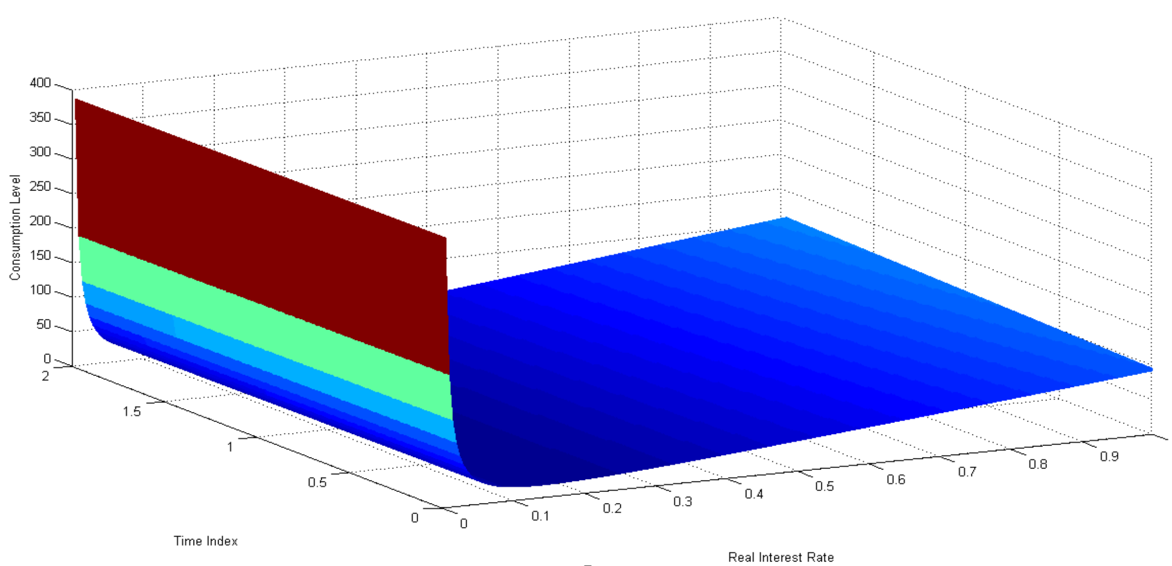
The solution for this problem, *i.e.*, the optimal consumption path is

$$c_t = rk_0 + \left(\frac{r-\rho}{\alpha}\right)\left\{t - \frac{1}{r}\right\}. \qquad (3.1.3)$$

In (3.1.3) consumption is based on default parameters, but time, t, which is selected by the consumer. As it can be seen, the optimal consumption path depends on ρ. For this reason it is interesting to modify the assumption that requires that this parameter is constant. In later models, this assumption will be relaxed, so that leads to new optimal consumption expressions, and thus new interpretations of the optimal paths. In order that the trajectory (3.1.3) is meaningful we should ask for

$$k_0 > (r-\rho)/\alpha r^2. \qquad (3.1.4)$$

This last constraint states that from $t = 0$ the consumer has a positive consumption path. Otherwise, if this inequality were strictly negative, it would imply that the endowment is insufficient to meet consumption throughout the time horizon, if equality is fulfilled, then consumption would be zero.



Graph 3.1. Consumption level of the conventional model as a function of r and t (Source: own elaboration).

Graph 3.1 shows the consumption path as a function of two parameters, *r* y *t*. For this example the following values are assigned: $k_0 = 100$, ρ = 0.2 y α = 0.1. While $r \in [0,1]$, where this interval represents a percentage scale; $t \in [0,2]$.

3.2 Subjective discount rate randomized (ITH)

Next, the parameter ρ is introduced in such a way that it has an a priori associated probability function, based on two arguments. The utility function is just as the previous model, negative exponential. Let assume that ρ is distributed exponentially. A first reason to choose this distribution is because describes an impatient behavior, *i.e.*, the density function is biased to the origin. The second reason consists on its operative easiness when manipulating equations.

$$u(c_t;\theta) = -e^{-\alpha c_t}, \quad \alpha > 0;$$
$$f_{\mathrm{P}}(\rho) = \beta e^{-\beta\rho}, \quad \beta > 0, \quad \rho \geq 0.$$

Then, the utility maximization problem is expressed as follows:

$$\begin{aligned} &\text{Maximize} \quad \int_0^\infty \left(\int_0^\infty -e^{-\alpha t} e^{-\rho t} dt \right) \beta e^{-\beta\rho} d\rho \\ &\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt} dt. \end{aligned}$$

Restriction conditions remain the same as the conventional model. The objective function is a double integral, where the outside integral corresponds in first instance to the values that can take ρ and the inner integral takes values that correspond to the time. The integral in brackets is nothing but the integral of the conventional model, the factor multiplying this integral corresponds to the exponential density function associated with the subjective discount rate.

Rewriting the above approach to perform some simplifications, it can be seen as:

$$\begin{aligned} &\text{Maximize} \quad \int_0^\infty \int_0^\infty -\beta e^{-\alpha c_t} e^{-\rho(t+\beta)} d\rho dt \\ &\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt} dt. \end{aligned} \qquad (3.2.1)$$

This new set up allows to apply the Fubini's Theorem, for an exchange of measures. Notice that it is possible to remove from the inner integral $-\beta e^{-\alpha c_t}$, since it does not depend of ρ, and solve the integral of $e^{-\rho(t+\beta)}$ with respect to ρ. The whole development of these simplifications is shown in appendix B.

Optimal consumption is obtained, again as before. It depends exclusively of known parameters for the consumer,

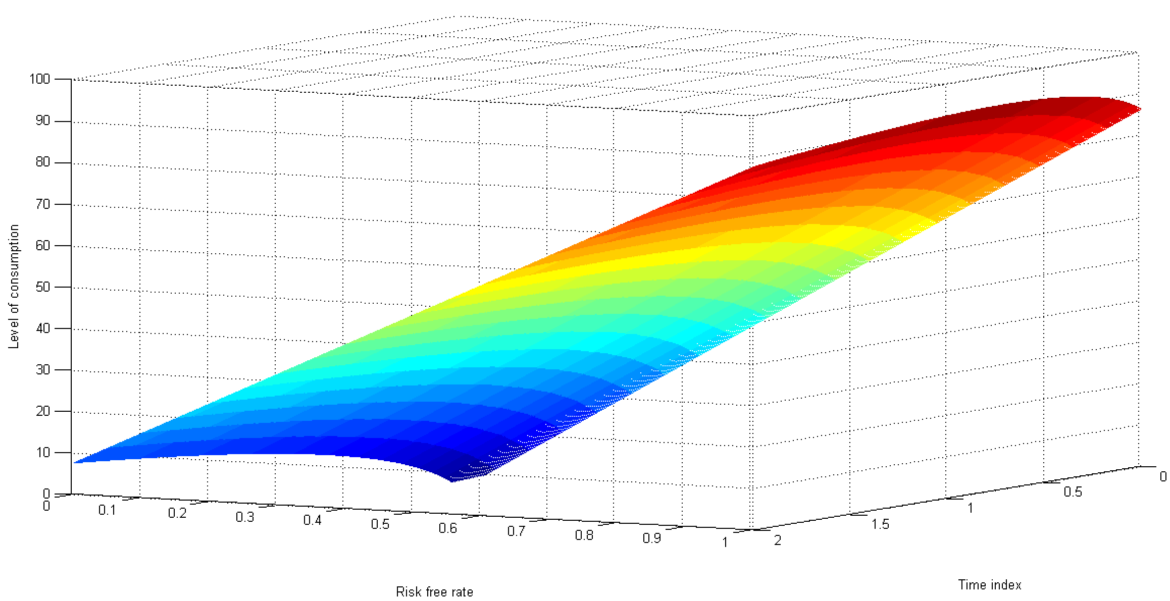
$$c_t = rk_0 + \frac{1}{\alpha}\left[\ln\left(\frac{1}{t+\beta}\right) + rt\right] - \frac{1}{\alpha}\left[1 + e^{r\beta}\left(C + \ln r\right)\right] \qquad (3.2.2).$$

This expression is more complex than (3.1.3), but it incorporates more information. As it was shown in the last section, the Lagrange multiplier is cleared with the purpose to determine it. Later, this value is substituted into the consumption function derived from the first order conditions. In (3.2.2) appears a Euler constant *C*, for more details about this number see Gradshteyn (2007).

The following condition must be accomplished when $t = 0$:

$$k_0 > \frac{1}{\alpha r}\left\{1 + \ln\beta + e^{r\beta}\left(C + \ln r\right)\right\}.$$

If the endowment is greater than the right hand side of inequality ensures that the consumer will have sufficient resources at every moment to reach her level of consumption. This model can be seen as a representative agent of a population where the number of individuals can change over time, but remains an exponential distribution representing her impatience with the current or future utility.



Graph 3.2. The level of consumption in the subjective discount rate as a random variable model as a function of *r* and *t* (Source: own elaboration).

Graph 3.2 shows the consumption level in the subjective discount rate randomized model as a function of *r* y *t*, with values: $k_0 = 100$, β = 0.2 y α = 0.1, while *r* y *t* keep the same values as in the graph 3.1.

3.3 Risk aversion rate randomized (ITH)

In the present section the risk aversion rate is considered a random variable. Assumptions are similar in this model to those exposed in section 3.2.

$$u(c_t;\theta) = -e^{-\alpha c_t}, \quad \alpha > 0;$$
$$f_{\mathrm{A}}(\alpha) = \mu e^{-\mu\alpha}, \quad \mu > 0.$$

So, the utility maximization problem is formally expressed as follows:

$$\text{Maximize} \quad \int_0^\infty \left(\int_0^\infty -e^{-\alpha c_t} e^{-\rho t} dt \right) \mu e^{-\mu\alpha} d\alpha$$
$$\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt} dt.$$

The inner integral coincides, again, with the objective function of the conventional model. It is necessary to include the density function of this variable into the problem since the risk aversion rate is considered randomized. Applying Fubini's theorem allows reorganizing the integrals:

$$\begin{aligned} &\text{Maximize} \quad \int_0^\infty \int_0^\infty -\mu e^{-\alpha(c_t+\mu)} e^{-\rho t} d\alpha dt \\ &\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt} dt. \end{aligned} \qquad (3.3.1)$$

Through some calculations, which are available in appendix C, it is obtained the optimal trajectory of the consumption.

$$c_t = e^{\frac{(r-\rho)}{2}t} \left[k_0 + \frac{\mu}{r} \right] \left(\frac{r+\rho}{2} \right) - \mu \qquad (3.3.2)$$

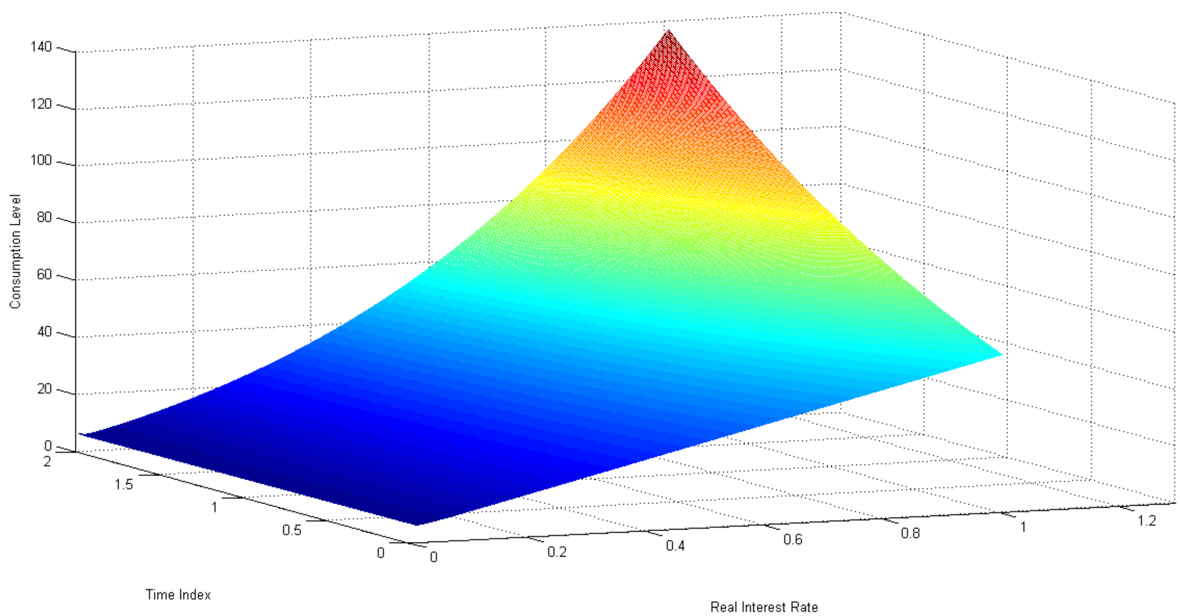
This equation is more complex than the conventional model, but it is less complicated than equation (3.2.2). Notice, once again, that the formula is closed and the parameters are all known. In order to prevent a negative value for the consumption is necessary that:

$$k_0 > \mu\left[\frac{2e^{-\frac{(r-\rho)}{2}t}}{r+\rho} - \frac{1}{r}\right].$$

So, at the beginning of the time horizon the initial stock must be greater than:

$$k_0 > \mu\left[\frac{2}{r+\rho} - \frac{1}{r}\right].$$

The last expression ensures that the consumer will be able to have enough resources at every point in time to satisfy her level of consumption. As in the previous model, the current section introduces the idea of an average consumer instead of a representative consumer. Heterogeneity is provided through the randomness of the risk aversion rate. This means that each member of the population assigns a different value for such rate, which is exponentially distributed.



Graph 3.3. Consumption level in the risk aversion randomized model as a function of r and t (Source: own elaboration).

Graph 3.3 illustrates the trajectories of the consumption as a function of the real interest rate and time. In this example the next values are considered: $k_0 = 100$, $\mu = 0.1$ y $\rho = 0.1$, $r \in [0,1]$ and $t \in [0,2]$.

3.4 Conventional model: finite time horizon (FTH)

The set up for this model is similar to section 3.1, the only difference is time horizon, which in this case is finite. Therefore, the discounted utility maximization problem facing the consumer, with a subjective discount rate ρ, is as follows:

$$\text{Maximize } \int_0^T u\left(c_t\right)e^{-\rho t}dt$$
$$\text{subject to } k_0 = \int_0^T c_t\, e^{-rt}dt. \qquad (3.4.1)$$

Where k_0 is the initial endowment which in this case is constant and c_t is the consumption at time t; the restriction in (3.4.1) indicates that the endowment and the discounted consumption, with a real interest rate, are the same.

Omitting the boundary conditions, Lagrangian is given by:

$$\mathfrak{L}\left(c_t,\lambda\right) = -e^{-\alpha c_t}e^{-\rho t} + \lambda\left(rk_0 - c_t\right)e^{-rt}$$

where λ is a constant to determine. Deriving with respect to c_t, it follows that

$$\alpha e^{-\alpha c_t}e^{-\rho t} - \lambda e^{-rt} = 0.$$

The above equation leads to

$$c_t = -\frac{1}{\alpha}\ln\left(\frac{\alpha}{\lambda}\right) + \frac{1}{\alpha}\left(r-\rho\right)t \qquad (3.4.2)$$

The last expression shows only known parameters a priori, but λ, which is the Lagrange's multiplier, and this must be isolated in order to obtain the optimal path through time. To find this value, the right side of the restriction in (3.4.1) is replaced by equation (3.4.2) is and the integral is solved. Once that the integral is solved λ is isolated and is substituted in (3.4.2). The development of the math is shown in appendix D.

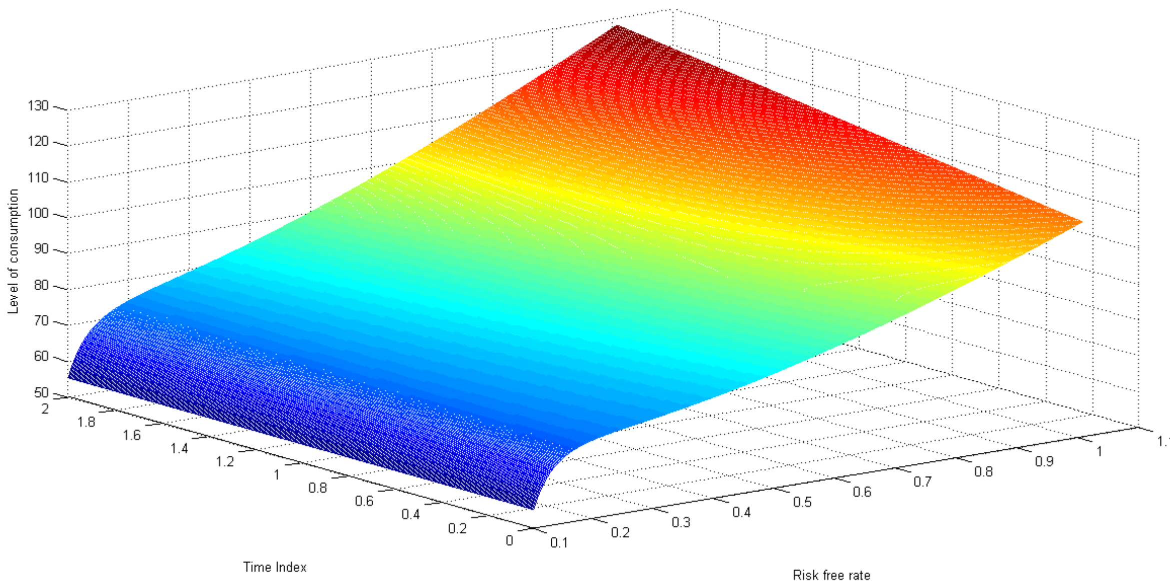
The solution for this problem, *i.e.*, the optimal consumption path is

$$c_t = \left\{rk_0 - \frac{\left(r-\rho\right)}{\alpha}\left[\frac{1}{r}\left[1-e^{-rT}\right] - Te^{-rT}\right]\right\}\left[1-e^{-rT}\right]^{-1} + \frac{1}{\alpha}\left(r-\rho\right)t. \qquad (3.4.3)$$

In (3.4.3), since all cases, consumption is based on default parameters, but time, t, which is selected by the consumer. As can be seen, the optimal consumption path depends on ρ. For this reason it is interesting to modify the assumption that requires that this parameter is constant. In later models, this assumption will be relaxed, so that leads to new optimal consumption expressions, and thus new interpretations of the optimal paths. For the trajectory in (3.4.3) is meaningful we should ask

$$k_0 > \frac{(r-\rho)}{\alpha r}\left\{\frac{1}{r}\left[1-e^{-rT}\right]-Te^{-rT}\right\}. \qquad (3.4.4)$$

This last constraint states that from the time $t = 0$ the consumer has a positive consumption path. Otherwise, if this inequality were strictly negative, it would imply that the provision is insufficient to meet consumption throughout the time horizon, if equality is fulfilled, then consumption would be zero.



Graph 3.4. Consumption level of the conventional model as a function of *r* and *t* (Source: own elaboration).

Graph 3.4 shows the consumption path as a function of two parameters, *r* y *t*. For this example the following values are assigned: $k_0 = 100$, T = 2, ρ = 0.1 y α = 0.1. While $r \in [0,1]$, where this interval represents a percentage scale; $t \in [0,2]$.

3.5 Subjective discount rate randomized (FTH)

Next, the parameter ρ is introduced in such a way that it has an a priori associated probability function. The utility function is just as the previous model, negative exponential. Let assume that ρ is distributed exponentially. The main reason to choose this distribution is merely because of its operative easiness when manipulating equations.

$$u(c_t;\theta) = -e^{-\alpha c_t}, \quad \alpha > 0;$$
$$f_{\mathrm{P}}(\rho) = \beta e^{-\beta\rho}, \quad \beta > 0, \quad \rho \geq 0.$$

Then, the utility maximization problem is expressed as follows:

$$\begin{aligned} &\text{Maximize} \quad \int_0^\infty \left(\int_0^T -e^{-\alpha t} e^{-\rho t} dt \right) \beta e^{-\beta\rho} d\rho \\ &\text{subject to} \quad k_0 = \int_0^T c_t e^{-rt} dt. \end{aligned}$$

Restriction conditions remain the same as the conventional model. The objective function is a double integral, where the outside integral corresponds in first instance to the values that can take ρ and the inner integral takes values that correspond to the time. The integral in brackets is nothing but the integral of the conventional model, the factor multiplying this integral corresponds to the exponential density function associated with the subjective discount rate.

Rewriting the above approach to perform some simplifications, it can be seen as:

$$\begin{aligned} &\text{Maximize} \quad \int_0^T \int_0^\infty -\beta e^{-\alpha c_t} e^{-\rho(t+\beta)} d\rho dt \\ &\text{subject to} \quad k_0 = \int_0^T c_t e^{-rt} dt. \end{aligned} \tag{3.5.1}$$

This new set up allows to apply the Fubini's Theorem, for an exchange of measures. Notice that it is possible to remove from the inner integral $-\beta e^{-\alpha c_t}$, since it does not depend of ρ, and solve the integral of $e^{-\rho(t+\beta)}$ with respect to ρ. The whole development of these simplifications is shown in appendix E.

Optimal consumption is obtained, again as before. It depends exclusively of known parameters for the consumer,

$$\begin{aligned} c_t = &\left\{ r\alpha k_0 - \left[1 - e^{-rT}(1 + rT)\right] - e^{r\beta}\left[C + \ln r + \ln(T+\beta) e^{-r(T+\beta)}\right]\right\}\left[\alpha\left(1 - e^{-rT}\right)\right]^{-1} \\ &+ \frac{1}{\alpha} rt + \frac{1}{\alpha} \ln\left(\frac{1}{t+\beta}\right). \end{aligned} \tag{3.5.2}$$

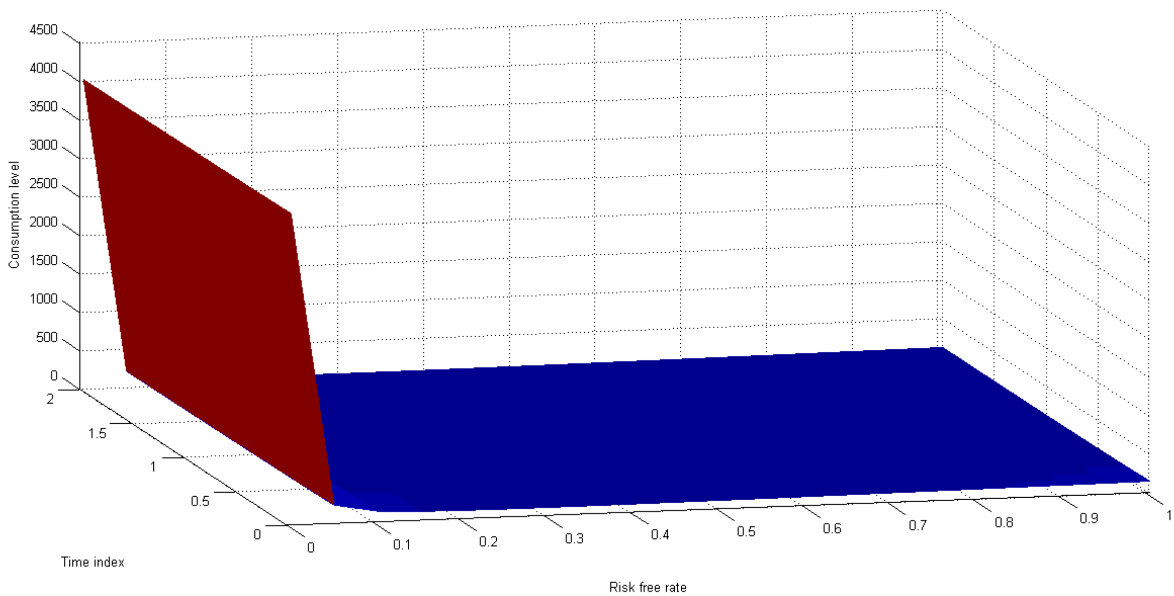
.

This expression is more complex than (3.5.3), but it incorporates more information. In order to get (3.5.2) a Lagrangian is proposed (see appendix 2). As it was shown in the last section, the Lagrange multiplier is cleared with the purpose to determine it. Later, this value is substituted in the consumption function derived from the first order conditions.

In equation (3.5.2) C represents the Euler constant. The following condition must be accomplished:

$$k_0 > \frac{1}{\alpha r}\left\{1+\ln(\beta)\left[1-e^{-rT}\right]-e^{-rT}(1+rT)-e^{r\beta}\left[C+\ln r+\ln(T+\beta)e^{-r(T+\beta)}\right]\right\}.$$

If the endowment is greater than the right side of inequality ensures that the consumer will have sufficient resources at every moment to meet her level of consumption. This model can be seen as a representative agent of a population where the number of individuals can change over time, but remains an exponential distribution representing her impatience with the current or future utility.



Graph 3.5. The level of consunption in the subjective discount rate as a random variable model as a function of *r* and *t* (Source: own elaboration).

Graph 3.5 shows the consumption as a function of *r* y *t*. For this example the following values are assigned: $k_0 = 100$, $T = 2$, $\beta = 0.2$ y $\alpha = 0.1$. While $r \in [0.05, 1]$ and $t \in [0, 2]$.

3.6 Risk aversion rate randomized (FTH)

This chapter finishes introducing the risk aversion rate as a random variable in a finite horizon set up. In this setting the assumptions are similar to those exposed earlier, for the other models developed so far in this chapter. So, once again the utility functions is exponential with a randomized parameter α,

$$u(c_t;\theta) = -e^{-\alpha c_t}, \quad \alpha > 0;$$
$$f_A(\alpha) = \mu e^{-\mu\alpha}, \quad \alpha > 0.$$

Accordingly, the maximization problem of a decision maker is as follows:

$$\text{Maximize} \quad \int_0^\infty \left(\int_0^T e^{-\alpha t} e^{-\rho t} dt \right) \mu e^{-\mu \alpha} d\,\alpha$$
$$\text{subject to} \quad k_0 = \int_0^T c_t e^{-rt} dt.$$

This looks like almost as the maximization problem in 3.3. However, a change in the time horizon modifies the solution, as it is expected. It is useful to do this exercise because if our result is true correct for the finite time horizon case, then taking the limit of *T* when it goes to an infinite number must give us the same solution obtained in section 3.3. In other words, solving the finite time horizon problem is equivalent to solve the infinite time horizon setting. Nevertheless, in terms of the details it is interesting to analyze both term structures. Proceeding as usual,

$$\begin{aligned} &\text{Maximize} \quad \int_0^T \int_0^\infty -\mu e^{-\alpha(c_t+\mu)} e^{-\rho t} d\alpha dt \\ &\text{subject to} \quad k_0 = \int_0^T c_t e^{-rt} dt. \end{aligned} \qquad (3.6.1)$$

It is necessary to emphasize that without Fubini's theorem our problem gets complicated. Fortunately, in every model proposed in this work is possible to apply it. When doing this is easier to deal with the integrals and this facilitates calculations. Obviously, the inner integral is solved and as a consequence the objective and the restriction have the same interval with respect to their respective integrals. The latter implies that the Lagrange's method of optimization is suitable for solving this simple decision problem.

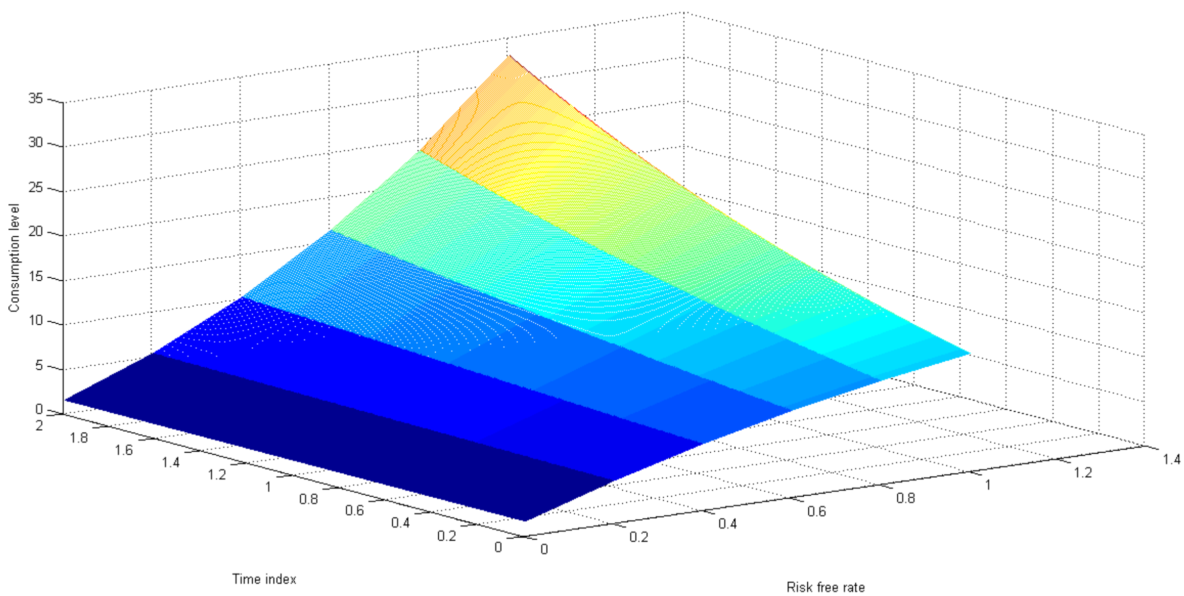
$$c_t = \left\{ k_0 + \frac{\mu}{r}\left[1 - e^{-rT}\right] \right\} \left(\frac{r+\rho}{2} \right) \left[1 - e^{-\frac{r-\rho}{2}T} \right]^{-1} e^{\frac{r-\rho}{2}t} - \mu. \qquad (3.6.2)$$

Details of the solution for the consumption path are available in the appendix F. In equation (3.6.2) appears parameter *T*, this represents a difference when comparing with

(3.3.2). As before, the initial endowment of capital must be strictly larger than zero at the beginning of the time period, this means

$$k_0 > \left(\frac{2\mu}{r+\rho}\right)\left[1-e^{-\frac{r-\rho}{2}T}\right]-\frac{\mu}{r}\left[1-e^{-rT}\right].$$

This condition guarantees a positive consumption through the period established in the maximization problem. With this in mind, the decision maker has achieved a primary goal: consumers in this economy always consume something. This last result is meaningful because it says that in a population where individuals differ in their risk aversion rate an optimal consumption is attained in such a way that in every instant of time during the finite horizon consumers in this economy always consume, and this consumption is neither negative nor zero.



Graph 3.6. The consumption level in the risk aversión rate as a random variable model as a function of *r* and *t* (Source: Own elaboration).

Last paragraph could be understood by means of graph 3.6, which shows the trajectory of the level of consumption as a function *r* and *t*. Values assigned to the parameters are: $k_0 = 100$, $T = 2$, $\mu = 0.1$, $\rho = 0.1$, $r \in [0,1]$ and $t \in [0,2]$.

CHAPTER 4. AVERAGE CONSUMER DECISIONS IN AN ECONOMY WITH HETEROGENEOUS SUBJECTIVE DISCOUNT RATES AND RISK AVERSION COEFFICIENTS: THE FINITE HORIZON CASE

This chapter aims to study the behavior of the average rational consumer of an economy populated by heterogeneous agents in a finite horizon framework. Heterogeneity takes into account both the subjective discount rate and risk aversion coefficient. Closed-form solutions for the optimal paths of consumption and capital, of the average consumer, are derived. Moreover, a closed form solution for the economic welfare of the average consumer is obtained. Finally, several analytical and graphical exercises of comparative statics are carried out.

4.1 Preference heterogeneity

Consider an economy where individuals are rational consumers endowed with the negative exponential function. This economy consumes and produces a single perishable good, and is populated by heterogeneous agents. Heterogeneity is represented by two distribution functions. The first distribution $F = F(\rho), \quad \rho > 0,$ is associated with the subjective discount rate, ρ, The second distribution $G = G(\alpha), \quad \alpha > 0,$ is related to the parameter α appearing in the negative exponential utility function $u(c_t;\alpha) = -e^{-\alpha c_t}$. It is reasonable to assume stochastic independence between ρ and α since anxiety for present consumption is not related to risk aversion. In what follows, it will be assumed that ρ and α are both driven by the exponential distribution, that is to say, the parameters α and ρ have, respectively, densities $g(\alpha) = -\mu e^{-\alpha\mu}$, $\mu > 0$, and $f(\rho) = -\lambda e^{-\lambda\rho}$, $\lambda > 0$, where both μ and λ are known parameters.

4.2 Resource allocation

Next, it is assumed that resource allocation in the economy is given by the national income identity and not by a price system. For the sake of simplicity, it will be assumed a closed economy without government, *i.e.*, a closed autarky. Suppose also that the rate of depreciation of capital is zero, thus the per capita national income identity satisfies

$$f(k_t) = c_t + \dot{k}_t$$

where k_t is capital, $f(k_t)$ is the production function, and c_t is consumption; all of them in per capita terms.

4.3 Firms' behavior

It is assumed that production is carried out by a representative firm using an "*Ak*" technology, *i.e.*, $y_t = f(k_t) = Ak_t$. The present value, *PV*, of the representative firm is given by:

$$PV = \int_0^T (Ak_t - rk_t) e^{-rt} \mathrm{d}\, t$$

where the difference in the integral is nothing more than the income of the firm less the payment to factor; in this case there is only payment to capital. It is worth noting that the above expression represents the benefits of the firm discounted with the real interest rate. The first order condition of the maximization problem of the representative firm leads to $r = A$. Thus, the marginal product of capital satisfies that the technological level is constant and equal to the real interest rate. Thus, after discounting and taking the present value of both sides of the per capita national income identity, and considering a finite transversality condition, it follows that

$$0 = \int_0^T c_s e^{-rs} \mathrm{d}s + \lim_{t \to T} k_t e^{-rt} - k_0,$$

or $k_0 = \int_0^T c_s e^{-rs} \mathrm{d}s,$ where k_0 is given.

4.4 Central planner's problem

It is assumed that a central planner wishes to maximize the consumption satisfaction of the average agent. Specifically, the central planner wishes to solve

$$\begin{aligned}&\text{Maximize} \quad \int_0^\infty\left(\int_0^\infty\left(\int_0^T -e^{-\alpha c_t}e^{-\rho t}\mathrm{d}t\right)\mu e^{-\mu\alpha}\mathrm{d}\alpha\right)\lambda e^{-\lambda\rho}\mathrm{d}\rho \\ &\text{subject to} \quad k_0=\int_0^T c_t e^{-rt}\mathrm{d}t.\end{aligned} \tag{1}$$

The necessary condition for the optimization problem stated in (1) is obtained as follows. Applying twice Fubini's theorem to the objective function, it follows

$$\int_0^T\int_0^\infty -\mu\lambda e^{-\rho(t+\lambda)}\int_0^\infty e^{-\alpha(c_t+\mu)}\mathrm{d}\alpha\ \mathrm{d}\rho\ \mathrm{d}t=\int_0^T\int_0^\infty \frac{-\mu\lambda e^{-\rho(t+\lambda)}}{c_t+\mu}\,\mathrm{d}\rho\ \mathrm{d}t$$

$$=\int_0^T\frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)}\mathrm{d}t.$$

Hence, the new optimization problem becomes:

$$\begin{aligned}&\text{Maximize} \quad \int_0^T\frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)}\mathrm{d}t\\ &\text{subject to} \quad \int_0^T\left(rk_t-\dot{k}_t\right)e^{-rt}\mathrm{d}t=\int_0^T c_t e^{-rt}\mathrm{d}t\end{aligned}$$

In this case, the Lagrangian is given by

$$\mathfrak{L}(c_t,\lambda)=\frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)}+\beta\left(rk_t-\dot{k}_t-c_t\right)e^{-rt}$$

where β is the Lagrange multiplier. Differentiating now with respect to c_t, it follows that

$$\frac{\partial\mathfrak{L}}{\partial c_t}=\frac{\mu\lambda}{(t+\lambda)(c_t+\mu)^2}-\beta e^{-rt}=0\,.$$

And after solving for c_t, it is obtained that

$$c_t=\sqrt{\frac{\mu\lambda}{\beta}\frac{e^{rt}}{(t+\lambda)}}-\mu\ .$$

By substituting the above expression in the constraint, it is derived that

$$k_0=\int_0^T\sqrt{\frac{\mu\lambda}{\beta}\frac{e^{rt}}{(t+\lambda)}}e^{-rt}\mathrm{d}t-\int_0^T\mu e^{-rt}\mathrm{d}t$$

$$k_0=\sqrt{\frac{\mu\lambda}{\beta}}\int_0^T\frac{1}{(t+\lambda)^{1/2}}e^{-\frac{r}{2}t}\mathrm{d}t-\frac{\mu}{r}\left(1-e^{-rT}\right)$$

$$\int_0^T\frac{1}{(t+\lambda)^{1/2}}e^{-\frac{r}{2}t}\mathrm{d}t=2\int_0^T\frac{1}{2(t+\lambda)^{1/2}}e^{-\frac{r}{2}t}\mathrm{d}t$$

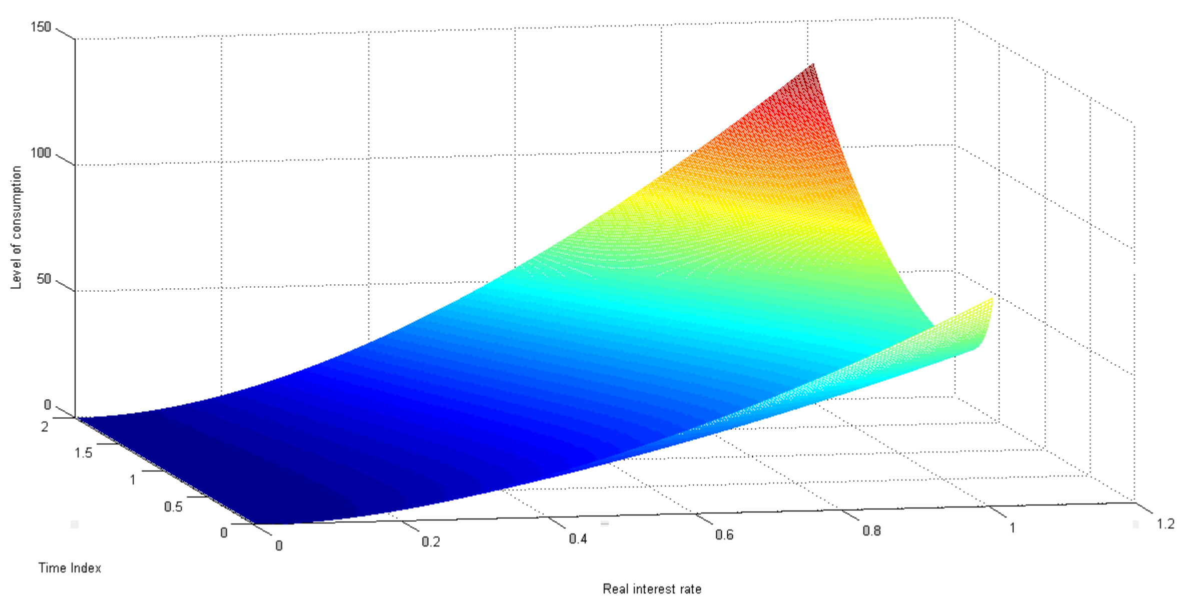
In the above equation, as usual, the Lagrange multiplier, β, is unknown. In order to find it, equation (2) is substituted into the constraint in (1). The optimal consumption path satisfies

$$c_t = \frac{e^{\frac{r}{2}(t-\lambda)}}{\sqrt{t+\lambda}}\left[\frac{\left(Ak_0+\mu\left(1-e^{-rT}\right)\right)}{\sqrt{8A\pi}\left(\Phi\left(\sqrt{r\left(T+\lambda\right)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)}\right]-\mu. \quad (3)$$

where Φ represents the cumulative distribution function of the standard normal random variable, and, as before, $r = A$ (computations are shown in Appendix G). Notice now that at time $t = 0$,

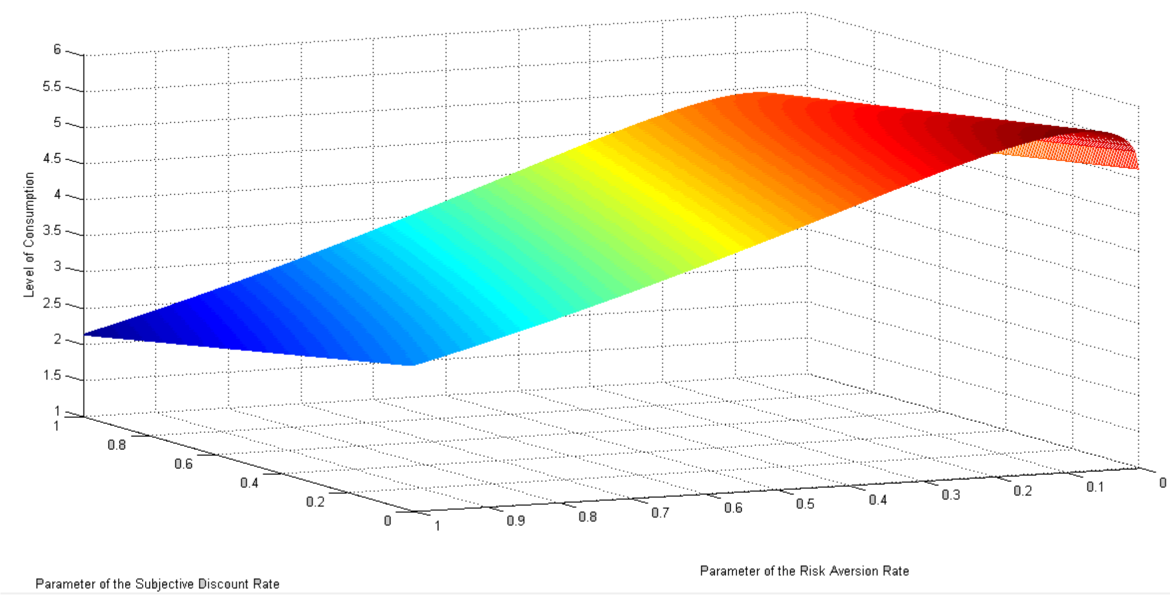
$$\frac{e^{-\frac{r\lambda}{2}}}{\sqrt{\lambda}}\left[\frac{\left(Ak_0+\mu\left(1-e^{-rT}\right)\right)}{\sqrt{8A\pi\left(\Phi\left(\sqrt{r\left(T+\lambda\right)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)}}\right] > \mu. \quad (4)$$

The left side of (4) is always positive, so c_t will also be positive for all $t \in [0,T]$. In order to carry out a graphic comparative statics exercise, it is illustrated in Graph 4.1 the path of optimal consumption, for the average agent, as a function of A and t, with all other parameters remaining constant. In this case is was supposed that $k_0 = 100$, λ = 0.1, α = 0.05, and $r \in \left(0,1\right]$. It is worth pointing out that consumption increases when both r and t rise.



Graph 4.1. The level of consumption as a function of r and t (Source: own elaboration).

Moreover, Graph 4.2 shows the behavior of the optimal path of consumption as a function of the risk aversion parameter and the subjective discount rate. In this case it was supposed that $\lambda \in (0,1]$ and $\mu \in (0,1]$. It is important to observe that consumption increases for very small values (less than 0.1) of the risk aversion parameter, after that value consumption decreases.



Graph 4.2. The level of consumption as a function of the risk aversion parameter and the subjective discount rate (Source: own elaboration).

Next, we carry out a comparative statics exercise by computing the derivatives of the consumption with respect to each parameter. Thus,

$$\frac{\partial c_t}{\partial \mu} = \frac{e^{\frac{r}{2}(t-\lambda)}}{\sqrt{8\pi r(t+\lambda)}}\left[\frac{1-e^{-rT}}{\Phi\left(\sqrt{r(T+\lambda)}-\Phi\sqrt{r\lambda}\right)}\right]-1>0,$$

$$\frac{\partial c_t}{\partial k_0} = \frac{\sqrt{r}\,e^{\frac{r}{2}(t-\lambda)}}{\sqrt{8\pi(t+\lambda)}}\left[\frac{1}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\right]>0,$$

$$\frac{\partial c_t}{\partial t} = \left[\frac{rk_0+\mu\left(1-e^{-rT}\right)}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\right]\frac{e^{\frac{r}{2}(t-\lambda)}}{2\sqrt{t+\lambda}}\left\{\frac{\sqrt{r}}{\sqrt{8\pi}}-\frac{1}{\sqrt{t+\lambda}}\right\}.$$

The sign of the latter partial derivative depends on the sign of the difference $\sqrt{r(t+\lambda)}-\sqrt{8\pi}$. Moreover, the following partial derivatives have ambiguous signs

$$\frac{\partial c_t}{\partial \lambda}=-\left[\frac{rk_0+\mu\left(1-e^{-rT}\right)}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\right]\left\{\frac{\sqrt{r}\,e^{\frac{r}{2}(t-\lambda)}}{2\sqrt{8\pi(t+\lambda)}}+\frac{e^{\frac{r}{2}(t-\lambda)}}{2(t+\lambda)}\right\}-\frac{e^{\frac{r}{2}(t-\lambda)}}{8\pi\sqrt{t+\lambda}}\frac{\left[rk_0+\mu\left(1-e^{-rT}\right)\right]}{\left[\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]^2}\left\{\frac{e^{-\frac{r}{2}(T+\lambda)}}{\sqrt{T+\lambda}}-\frac{e^{-\frac{r}{2}\lambda}}{\sqrt{\lambda}}\right\},$$

$$\frac{\partial c_t}{\partial T}=\frac{\sqrt{r}e^{\frac{r}{2}(t-\lambda)}}{\left[\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]\sqrt{8\pi r(t+\lambda)}}\left\{\mu\sqrt{r}\,e^{-rT}-\frac{\left[rk_0+\mu\left(1-e^{-rT}\right)\right]}{\left[\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]}\frac{e^{-\frac{r}{2}(T+\lambda)}}{\sqrt{8\pi(T+\lambda)}}\right\}.$$

Finally, it readily follows that

$$\frac{\partial c_t}{\partial r}=\frac{e^{\frac{r}{2}(t-\lambda)}}{\sqrt{t+\lambda}}\left[\frac{rk_0+\mu\left(1-e^{-rT}\right)}{\sqrt{8\pi r}\left[\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]}\right]\left\{\frac{t-\lambda}{2}+\frac{1}{2r}+\frac{e^{-\frac{r}{2}\lambda}}{\sqrt{8\pi r}}\left\{\sqrt{T+\lambda}\,e^{-\frac{r}{2}T}-\sqrt{\lambda}\right\}\right\}$$
$$+\frac{e^{\frac{r}{2}(t-\lambda)}}{\sqrt{t+\lambda}}\left[\frac{k_0+\mu Te^{-rT}}{\sqrt{8\pi r}\left[\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]}\right]>0.$$

Notice also that $\partial c_t/\partial k_0>0,$ that is, the level of consumption increases when the initial level of stock increases. Moreover, $\partial c_t/\partial\mu>0,$ that is, an increase in the risk aversion rate parameter positively affects the level of consumption. It is important to point out that the sign of $\partial c_t/\partial t$ depends on the sign of the difference $\sqrt{r(t+\lambda)}-\sqrt{8\pi}$ **.** Unfortunately, $\partial c_t/\partial\lambda$ and $\partial c_t/\partial T$ have ambiguous signs. Finally, $\partial c_t/\partial r>0,$ hence if the real interest rises, the level of consumption increases.

On the other hand, by substituting optimal consumption of the average individual in the national income identity, it follows that (see Appendix H)

$$k_t=e^{rt}\left[k_0+\frac{\mu}{r}\right]-\frac{\mu}{r}-\frac{1}{r}e^{rt}\left[\frac{\left(rk_0+\mu\left(1-e^{-rT}\right)\right)}{\left(\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)}\right]\left[\Phi\left(\sqrt{r(t+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right].$$

It is also interesting to compute the partial derivatives of capital. First, note that

$$\frac{\partial k_t}{\partial k_0}=e^{rt}\left\{1-\frac{\Phi\left(\sqrt{r(t+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\right\}>0,\quad \text{then}\quad \lim_{t\to T}\frac{\partial k_t}{\partial k_0}=0.$$

Observe now that

$$\frac{\partial k_t}{\partial T}=-\frac{e^{rt}}{r}\left[\Phi\left(\sqrt{r(t+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]\left\{\frac{\mu r e^{-rT}}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}-\frac{\sqrt{r}e^{-\frac{r}{2}(T+\lambda)}\left[rk_0+\mu\left(1-e^{-rT}\right)\right]}{\sqrt{8\pi(T+\lambda)}\left[\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]^2}\right\}.$$

Thus, the sign of the above partial derivative depends of the magnitude of the summands in the braces. Moreover,

$$\frac{\partial k_t}{\partial \mu}=\frac{e^{rt}}{r}\left\{1-\left[\frac{1-e^{-rT}}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\right]\left[\Phi\left(\sqrt{r(t+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]\right\}-\frac{1}{r},\ \text{then}\ \lim_{t\to T}\frac{\partial k_t}{\partial \mu}=0.$$

Also,

$$\frac{\partial k_t}{\partial \lambda}=-\frac{e^{rt}}{\sqrt{8\pi r}}\frac{\left(rk_0+\mu\left(1-e^{-rT}\right)\right)}{\left[\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]}\left\{\left(\frac{e^{-\frac{r}{2}(t+\lambda)}}{\sqrt{t+\lambda}}-\frac{e^{-\frac{r}{2}\lambda}}{\sqrt{\lambda}}\right)-\frac{\Phi\left(\sqrt{r(t+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\left(\frac{e^{-\frac{r}{2}(T+\lambda)}}{\sqrt{T+\lambda}}-\frac{e^{-\frac{r}{2}\lambda}}{\sqrt{\lambda}}\right)\right\},$$

then $\lim_{t\to T}\frac{\partial k_t}{\partial \lambda}=0.$

Moreover,

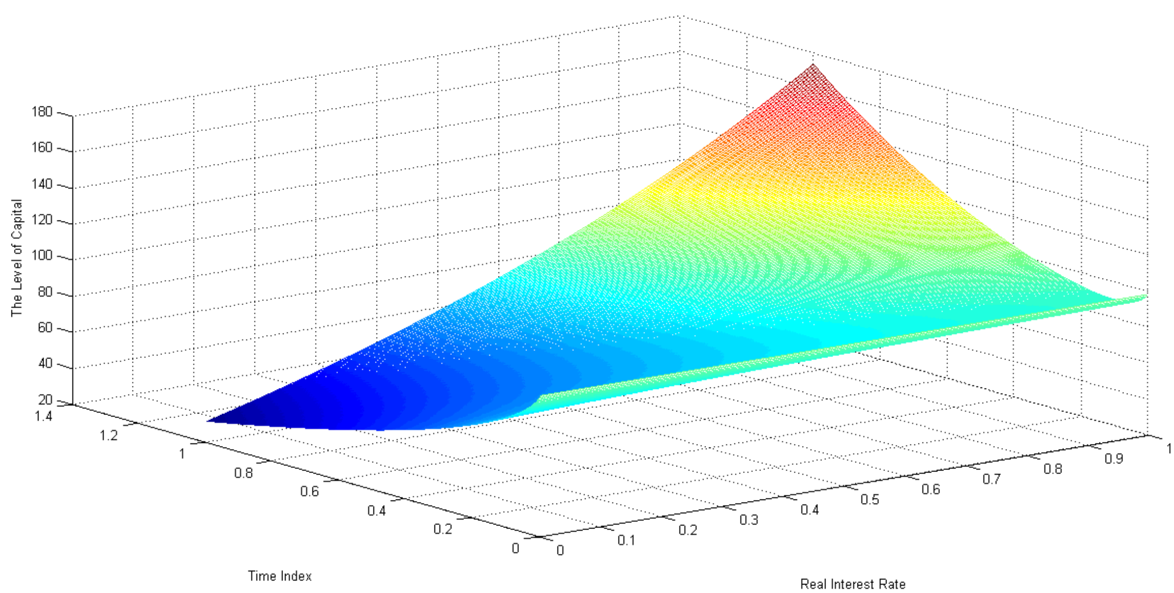
$$\begin{aligned}\frac{\partial k_t}{\partial r}&=\frac{\mu}{r^2}\left(1-e^{rt}\right)+\frac{1}{t}\left[k_0+\frac{\mu}{r}\right]e^{rt}+\left[\frac{rk_0+\mu\left(1-e^{-rT}\right)}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\right]\left[\Phi\left(\sqrt{r(t+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]\left\{\frac{e^{rt}}{r^2}-\frac{e^{rt}}{rt}\right\}\\&-\frac{e^{rt}}{r}\left[\frac{k_0+\mu T e^{-rT}}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\right]\left[\Phi\left(\sqrt{r(t+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]\\&-e^{rt}\frac{e^{-\frac{r\lambda}{2}}}{\sqrt{8\pi r}}\left(\frac{rk_0+\mu\left(1-e^{-rT}\right)}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\right)\left\{\frac{\Phi\left(\sqrt{r(t+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\left[\frac{e^{-\frac{rT}{2}}}{\sqrt{T+\lambda}}-\frac{1}{\sqrt{\lambda}}\right]-\left[\frac{e^{-\frac{rt}{2}}}{\sqrt{T+\lambda}}-\frac{1}{\sqrt{\lambda}}\right]\right\}\end{aligned}$$

has ambiguous sign. Finally,

$$\begin{aligned}\frac{\partial k_t}{\partial t}&=\frac{e^{rt}}{r}\left[k_0+\frac{\mu}{r}\right]-\frac{e^{rt}}{r^2}\left[\frac{rk_0+\mu\left(1-e^{-rT}\right)}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\right]\left[\Phi\left(\sqrt{r(t+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]\\&-e^{rt}\frac{e^{-\frac{r\lambda}{2}}}{\sqrt{8\pi r}}\left(\frac{rk_0+\mu\left(1-e^{-rT}\right)}{\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)}\right)\left[\frac{e^{-\frac{rt}{2}}}{\sqrt{T+\lambda}}-\frac{1}{\sqrt{\lambda}}\right]>0.\end{aligned}$$

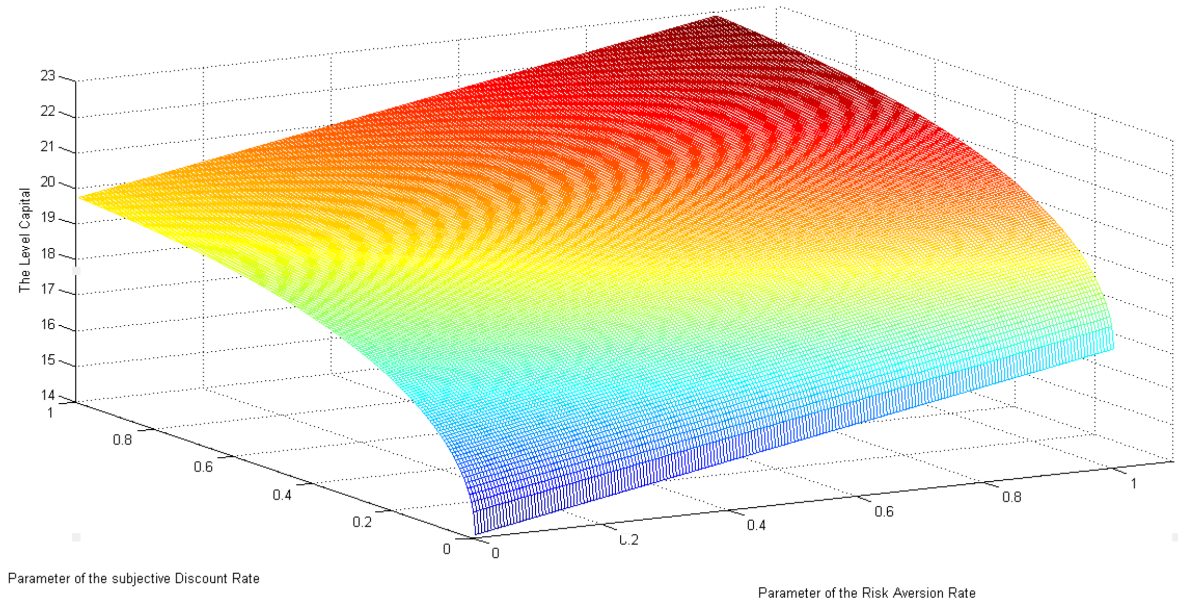
After computing the partial derivatives of k_t with respect to other variables, it is observed that: $\partial k_t/\partial k_0 > 0$, just as in the case of consumption, an increase in initial capital makes the level of capital increases; $\partial k_t/\partial \mu = \partial k_t/\partial \lambda = 0$ when $t \to T$, this implies that neither the risk aversion parameter nor the subjective discount rate parameter affect the level of capital when *t* approaches to *T*; $\partial k_t/\partial t < 0$, this means that a change in *t* decreases the level of capital stock; and $\partial k_t/\partial T$ and $\partial k_t/\partial r$ have both ambiguous sign.

Graph 4.3 illustrates the path of capital, for the average consumer, as a function of *r* and *t*; all other parameters remaining constant. For illustrative purposes, it is assumed that $k_0 = 100$, λ = 0.1, α = 0.05, and $r \in (0,1]$. In this case, that capital increases when both *r* and *t* rise.



Graph 4.3. The level of capital as a function of *r* and *t* (Source: own elaboration).

Finally, Graph 4.4 shows the behavior of the optimal path of capital when $\lambda \in (0,1]$ and $\mu \in (0,1]$. Notice, as expected, that capital increases when both the subjective discount rates and the risk aversion parameter.



Graph 4.4. The level of capital as a function of the risk aversion parameter and the subjective discount rate (Source: own elaboration).

4.5 Economic welfare of the average consumer

In what follows, the indirect utility or economic welfare function of the average consumer, *W*, will be computed. By substituting $c_t + \mu$ in the expected total utility in (1), it is found that

$$W = \int_0^T \frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)} \mathrm{d}t = \frac{-(\mu\lambda)e^{\frac{r\lambda}{2}}\sqrt{8A\pi}\left(\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)}{rk_0+\mu\left(1-e^{-rT}\right)}\int_0^T \frac{e^{-\frac{rs}{2}}}{(s+\lambda)^{\frac{1}{2}}}\mathrm{ds}$$

$$= \frac{-8\pi(\mu\lambda)e^{r\lambda}}{rk_0+\mu\left(1-e^{-rT}\right)}\left[\Phi\left(\sqrt{T+\lambda}\right)-\Phi\left(\sqrt{\lambda}\right)\right]\left[\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right].$$

This is a closed formula when all the parameters are known (details are available in Appendix I). In this case, it can be shown that

$$\frac{\partial W}{\partial k_0} > 0, \quad \frac{\partial W}{\partial \mu} < 0, \quad \frac{\partial W}{\partial T} > 0, \quad \frac{\partial W}{\partial \lambda} < 0, \quad \text{and} \quad \frac{\partial W}{\partial r} > 0 \text{ or } \frac{\partial W}{\partial r} < 0.$$

The first derivative is almost intuitive and it means that if the average consumer increased his/her initial stock of capital, then the welfare would increase. In a similar way, when the time horizon is extended, $t \rightarrow T$, the welfare function augments its value. On the other hand, there exists a negative relation between the preference parameters and the welfare function. Finally, the relation between the risk free rate and the welfare function is ambiguous (see Appendix J).

CHAPTER 5. CONSUMPTION DECISIONS OF THE AVERAGE AGENT IN AN ECONOMY WITH HETEROGENEOUS PREFERENCES DEFINED BY A BIVARIATE DISTRIBUTION

Chapter 5 considers an economy populated by heterogeneous individuals in two respects: both parameters representing the subjective discount and the risk aversion rates are supposed to have a joint distribution. That is, consumers differ in their level of anxiety for present consumption and their risk aversion rate. The utility index is of the negative exponential type. This research provides a closed-form optimal consumption path of the average infinite-lived agent. Finally, some comparative statics experiments are carried out.

5.1 Assumptions of the economy

Consider an economy where individuals are rational. This economy consumes and produces a single perishable good and is populated by heterogeneous agents in preferences. The heterogeneity in tastes of agents is represented by two distribution functions. The first function models the subjective discount rate as a random variable (the agents are indexed by the parameter ρ), i.e. $F = F(\rho), \quad \rho > 0.$

By the second distribution function, the depth parameter (or the level of learning, see Venegas (2000)) α of the negative exponential utility function is considered a random variable, $G = G(\alpha), \quad \alpha > 0.$

This model allows different agents are assigned the same values of the parameters mentioned above. Therefore, the probability of randomly choosing two consumer groups with indexes in the subsets of positive real numbers S_ρ and S_α are given respectively by

$$P\{\rho \in S_\rho\} = \int_{S_\rho} \mathrm{d}F(\rho) \text{ and } P\{\alpha \in S_\alpha\} = \int_{S_\alpha} \mathrm{d}G(\alpha).$$

Independence between ρ and α is assumed; this in addition to being reasonable as anxiety about present or future consumption is not related to learning in consumption, it also simplifies the algebraic development.

5.2 Problem of the central planner

It is assumed then that there is a central planner who wishes to maximize the satisfaction of the average agent. Specifically, it is assumed that the center planner wishes to maximize the following objective function

$$U = \int_{\rho>0}^{\infty}\left(\int_{\alpha>0}^{\infty}\left(\int_{t=0}^{\infty} u\left(c_t;\alpha\right)e^{-\rho t}\mathrm{d}t\right)\mathrm{d}G\left(\alpha\right)\right)\mathrm{d}F\left(\rho\right), \qquad (1)$$

where $u\left(c_t;\alpha\right)$ is the utility function of a commodity of consumption per capita. It is also assumed that the functional form of the utility function is exponential negative, i.e.

$$u\left(c_t;\alpha\right) = -e^{-\alpha c_t}, \quad \alpha > 0.$$

You can verify that this function is concave, meaning that

$$u_c = \alpha e^{-\alpha c_t} > 0 \text{ and } u_{cc} = -\alpha^2 e^{-\alpha c_t} < 0.$$

Moreover, it is assumed that the distribution function of ρ is given by $F\left(\rho\right) = 1 - e^{-\lambda\rho}, \ \ \lambda > 0,$

similarly define the distribution function for α as

$$G\left(\alpha\right) = 1 - e^{-\alpha\rho}, \quad \alpha > 0.$$

Accordingly, $\mathrm{d}F\left(\rho\right) = \lambda e^{-\lambda\rho}\mathrm{d}\rho$ and $\mathrm{d}G\left(\alpha\right) = \mu e^{-\mu\alpha}\mathrm{d}\alpha$.

5. 3 Behavior of firms

It is assumed that the production is performed by representative firm using an Ak technology, i.e.

$$y_t = f\left(k_t\right) = Ak_t.$$

The present value, PV, of the representative firm is given by the following integral:

$$PV = \int_{t=0}^{\infty}\left(Ak_t - rk_t\right)e^{-rt}dt$$

where the difference in the integral is nothing more than the income of the firm less payment to factor, in this case there is only the payment to capital. Note that the above expression represents the discounted benefits of the company with the real interest rate. The first order condition of the maximization problem of the representative firm leads to $A = r$.

Thus, the marginal product of capital satisfies the technological level is constant and equal to the real interest rate.

5.4 Resource allocation

However, it is assumed that resource allocation is given by the national income identity (for a closed economy without government, i.e. autarky) and not by a price system, as in López-Herrera *et al* (2012), except that the rate of depreciation of capital is zero. Thus, $rk_t = c_t + \dot{k}_t$.

Discounting and integrating both sides and considering the transversality condition, it follows that

$$0 = \int_0^\infty c_s e^{-rs}\mathrm{d}s + \lim_{t\to\infty} k_t e^{-rt} - k_0.$$

Therefore, $k_0 = \int_0^\infty c_s e^{-rs}\mathrm{d}s.$

For developmental outcome is regarded that the income is fixed.

5.5 Optimal consumption paths of the average agent

The average consumer has infinite life. The initial endowment of consumer is constant over time. In short, the problem facing the consumer is described as:

$$\text{Maximize} \quad \int_0^\infty \left(\int_0^\infty \left(\int_0^\infty -e^{-\alpha c_t} e^{-\rho t}\mathrm{d}t \right) \mu e^{-\mu\alpha}\mathrm{d}\alpha \right) \lambda e^{-\lambda\rho}\mathrm{d}\rho$$

$$\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt}\mathrm{d}t.$$

Note that the above expression is indeed the equation (1). Both μ and λ are known parameters, the first of the level of learning and the second of the subjective discount rate. Simplifying the problem statement, given the above assumptions, and applying Fubini's theorem, of the exchange of measures, it can be rewritten as follows:

$$\text{Maximize} \quad \int_0^\infty \frac{-\mu\lambda}{(t+\lambda)(\mu + c_t)}\mathrm{d}t$$

$$\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt}\mathrm{d}t.$$

The Lagrangian for this approach is:

$$\mathfrak{L}(c_t,\lambda)=\frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)}+\beta(y-c_t)e^{-rt}.$$

Differentiating with respect to c_t

$$\mathfrak{L}_{c_t}=\frac{\mu\lambda}{(t+\lambda)(c_t+\mu)^2}-\beta e^{-rt}=0$$

and solving for c_t

$$c_t=\sqrt{\frac{\mu\lambda}{\beta}}\sqrt{\frac{e^{rt}}{(t+\lambda)}}-\mu. \tag{2}$$

From expression (2) is unknown as to what is the Lagrange multiplier, to find that value is replaced (2) in the restriction and integrates with respect to time. The complete development of the calculations is shown in the appendix. The solution for optimal consumption path is

$$c_t=\frac{e^{\frac{r}{2}(t-\lambda)}}{\sqrt{t+\lambda}}\left[\frac{rk_0+\mu}{\sqrt{8\pi r\left(1-\Phi\left(\sqrt{r\lambda}\right)\right)}}\right]-\mu. \tag{3}$$

where Φ represents the accumulation function of a standard normal distribution function (see the appendix). Since $A = r$, recall section 3, r can be replaced by A, such that

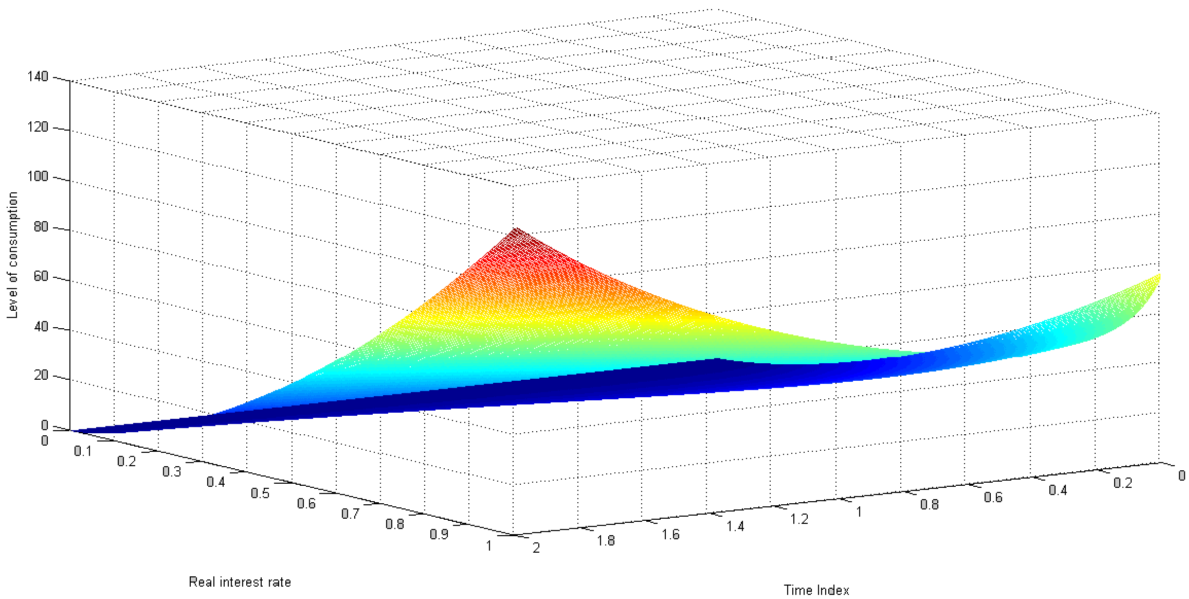
$$c_t=\frac{e^{\frac{A}{2}(t-\lambda)}}{\sqrt{t+\lambda}}\left[\frac{Ak_0+\mu}{\sqrt{8\pi A\left(1-\Phi\left(\sqrt{A\lambda}\right)\right)}}\right]-\mu.$$

In both expressions all the parameters are known, since the consumer decides the moment of time when to consume, then t is also known. The solution is well defined for all time, t, provided

$$\frac{e^{-\frac{r\lambda}{2}}}{\sqrt{\lambda}}\left[\frac{rk_0+\mu}{\sqrt{8\pi r\left(1-\Phi\left(\sqrt{r\lambda}\right)\right)}}\right]>\mu. \tag{4}$$

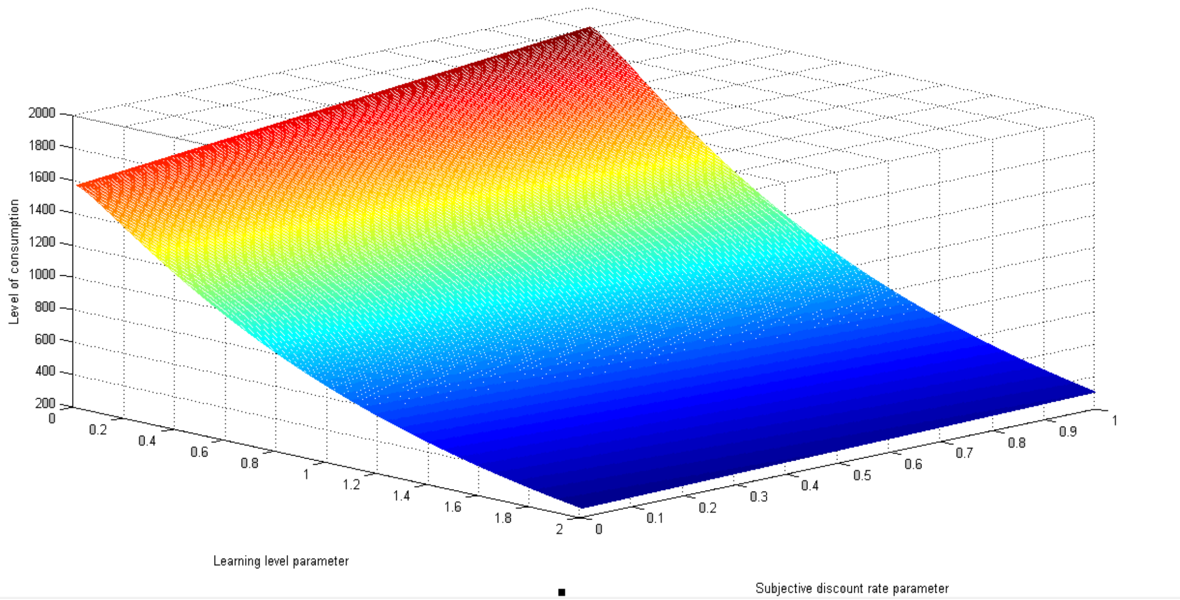
This result is due to assess the optimum consumption at time $t = 0$. Note that all factors in (4) are positive numbers. So, we have an expression valid for all t such that $t \in [0,\infty)$. One possible interpretation for (4) is that it is the average consumption of a population at time t, i.e. the optimal consumption path is represented as an expected value. In other words, this is the average consumption of a representative agent.

The following graph illustrates the trajectory of consumption as a function of two parameters, *r* and *t*, all other parameters behave as constants. For this example the values assigned are: $k_0 = 100, \lambda$ = 0.1 and α = 0.2. Whereas that $r \in (0,1]$; this range represents a percent, with minimum at 0% and a maximum at 10%.



Graph 5.1. The level of consumption as a function of *r* and *t* (own elaboration source).

Using this graph you can see that the level of consumption is increasing with respect to *r* and *t*. Moreover, also useful to observe the behavior of the trajectory of optimum consumption when *r* and *t* are fixed, whereas λ and α behave as variables that explain the different consumption levels. To show this case is considered an *r* = 5% and *t* = 10. The intervals associated with the subjective discount rate and the depth parameter take values $\lambda \in (0,1]$ and $\mu \in (0,2]$ respectively. Both intervals can be considered as rates expressed in percentages, as the real interest rate in graph 5.2.



Graph 5.2. Level of consumption as a function of the risk aversion rate ante the subjective discount rate. Own elaboration source.

Consumption levels, as before, is increasing with respect to parameters of the random variables. To generate both graphs is expressed in the condition (4).

Finally, by substituting optimal consumption of the average individual in the national income identity, it follows that

$$k_t = k_0 e^{rt} - e^{rt}\int_0^t c_s e^{-rs} ds =$$

$$= k_0 e^{rt} - e^{rt}\int_0^t \left\{ \frac{e^{\frac{r}{2}(s-\lambda)}}{\sqrt{s+\lambda}} \left[\frac{rk_0+\mu}{\sqrt{8\pi r\left(1-\Phi\left(\sqrt{r\lambda}\right)\right)}} \right] - \mu \right\} e^{-rs} \mathrm{d}s.$$

After some calculations,

$$k_t = e^{rt}\left[k_0 + \frac{\mu}{r}\right] - \frac{\mu}{r} - \left[\frac{rk_0+\mu}{\sqrt{8\pi r\left(1-\Phi\left(\sqrt{r\lambda}\right)\right)}} \right]\left[\left(2\sqrt{t+\lambda}-1\right)e^{\frac{r}{2}(t+\lambda)} - \left(2\sqrt{\lambda}-1\right)e^{-r\left(t-\frac{\lambda}{2}\right)}\right].$$

5.6 The welfare function of the average consumer

With some calculations it is possible to find the expression for welfare function, *W*. By substituting $c_t + \mu$ in the value function, it is found that

$$W = \int_0^\infty \frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)}\mathrm{d}t$$

$$= -\frac{1}{rk_0+\mu}(\mu\lambda)(8\pi)e^{r\lambda}\left[1-\Phi\left(\lambda^{1/2}\right)\right]\left\{r\left(1-\Phi\left([r\lambda]^{1/2}\right)\right)\right\}^{1/2}.$$

Fortunately, the expression above is a closed formula.

CONCLUSIONS

This research has focused upon the representation of a new alternative for the discounted utility model. Allowing both the intensity parameter for consumption and the risk aversion are random variables is a major challenge for the implementation of the problem, so for the solutions from it. Despite this, what is lost in simplicity is offset by the possibility of modeling in better shape intertemporal decisions of consumers. Moreover, it has been achieved finding closed-form formulas for both: infinite horizon and finite horizon.

Through these models it has been possible to represent an average consumer when the economy is populated by consumers with different tastes and preferences, representing an advance with respect to previous literature, where only the homogeneous case is considered. Certain functions, both distribution and utility functions, have been considered by their mathematical goodness. So the modification of these, and other, aspects enrich the existing literature as to possibly allow a reinterpretation of the problems of intertemporal utility maximization.

Possible applications of these proposals include the statistical analysis of individual consumption; broader extensions, and direct employment for specific situations such as insurance, retirement pensions and so on. Consider the factors that influence the parameters of the random variables raised here, they represent a source of enrichment of the literature in economics, and possibly in finance, by means of this new proposed approach to the problem of utility maximization.

APPENDIX A. CONVENTIONAL MODEL WITH INFINTE HORIZON

Next, the steps to get the result of the model are detailed:

$$\text{Maximize} \quad \int_0^\infty -e^{-\alpha c_t} e^{-\rho t} dt$$

$$\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt} dt.$$

$$\mathfrak{L}(c_t, \lambda) = -e^{-\alpha c_t} e^{-\rho t} + \lambda (k_0 - c_t) e^{-rt}$$

$$\mathfrak{L}_{c_t} = \alpha e^{-\alpha c_t} e^{-\rho t} - \lambda e^{-rt} = 0,$$

$$c_t = -\frac{1}{\alpha} \ln\left(\frac{\alpha}{\lambda}\right) + \frac{1}{\alpha}(r-\rho)t.$$

$$k_0 = \int_0^\infty \left(\frac{1}{\alpha} \ln\left(\frac{\alpha}{\lambda}\right) + \frac{1}{\alpha}(r-\rho)t\right) e^{-rt} dt,$$

$$k_0 = \frac{1}{\alpha r} \ln\left(\frac{\alpha}{\lambda}\right) + \frac{(r-\rho)}{\alpha r^2},$$

$$rk_0 = \frac{1}{\alpha} \ln\left(\frac{\alpha}{\lambda}\right) + \frac{(r-\rho)}{\alpha r}.$$

It follows that,

$$\frac{1}{\alpha} \ln\left(\frac{\alpha}{\lambda}\right) = rk_0 - \frac{(r-\rho)}{\alpha r}.$$

Replacing the last expression into c_t,

$$c_t = rk_0 - \frac{(r-\rho)}{\alpha r} + \frac{(r-\rho)}{\alpha} t,$$

$$c_t = rk_0 + \frac{(r-\rho)}{\alpha}\left(t - \frac{1}{r}\right).$$

APPENDIX B. THE MODEL OF THE SUBJECTIVE DISCOUNT RATE AS RANDOM VARIABLE WITH INFINITE HORIZON

For this setting the problem to solve is:

$$\text{Maximize} \quad \int_0^\infty -\beta e^{-\alpha c_t}\left(\int_0^\infty e^{-\rho(t+\beta)}d\rho\right)dt$$

$$\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt}dt$$

First, the inner integral is solved respect to the subjective discount rate. Then, the new problem is

$$\text{Maximize} \quad \int_0^\infty -\frac{\beta}{t+\beta}e^{-\alpha c_t}dt$$

$$\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt}dt$$

$$\mathcal{L}(c_t,\lambda) = \frac{-\beta e^{-\alpha c_t}}{(t+\beta)} + \lambda(k_0 - c_t)e^{-rt},$$

$$\mathcal{L}_{c_t} = \frac{\alpha\beta e^{-\alpha c_t}}{(t+\beta)} - \lambda e^{-rt} = 0,$$

$$\alpha c_t - rt = \ln\left(\frac{\alpha\beta}{\lambda(t+\beta)}\right),$$

$$c_t = \frac{1}{\alpha}\ln\left(\frac{\alpha\beta}{\lambda}\right) + \frac{1}{\alpha}\ln\left(\frac{1}{t+\beta}\right) + \frac{1}{\alpha}rt.$$

Substituting c_t into the restriction,

$$k_0 = \int_0^\infty\left(\frac{1}{\alpha}\ln\left(\frac{\alpha\beta}{\lambda}\right) + \frac{1}{\alpha}\ln\left(\frac{1}{t+\beta}\right) + \frac{1}{\alpha}rt\right)e^{-rt}dt.$$

$$\int_0^\infty \frac{1}{\alpha}\ln\left(\frac{\alpha\beta}{\lambda}\right)e^{-rt}dt = \frac{1}{\alpha r}\ln\left(\frac{\alpha\beta}{\lambda}\right),$$

$$\int_0^\infty \frac{1}{\alpha}rte^{-rt}dt = \frac{1}{\alpha r},$$

$$\int_0^\infty \frac{1}{\alpha}\ln\left(\frac{1}{t+\beta}\right)e^{-rt}dt = -\frac{1}{\alpha}\int_0^\infty \ln(t+\beta)e^{-rt}dt = -\frac{e^{r\beta}}{\alpha}\int_0^\infty \ln(t+\beta)e^{-r(t+\beta)}dt.$$

The former integral requires a little trick. A new function, which integrates the same value, is proposed

$$-\frac{e^{r\beta}}{\alpha}\int_0^{\infty}\ln(t+\beta)e^{-r(t+\beta)}dt=\begin{cases}-\frac{e^{r\beta}}{\alpha}\int_{-\beta}^{\infty}\ln(t+\beta)e^{-r(t+\beta)}dt & \text{if } t+\beta>0\\ 0 & \text{otherwise}\end{cases},$$

$$=\begin{cases}-\frac{e^{r\beta}}{\alpha}\int_0^{\infty}\ln(x)e^{-r(x)}dx & \text{if } t+\beta=x>0\\ 0 & \text{otherwise}\end{cases},$$

$$=\begin{cases}\frac{e^{r\beta}}{\alpha r}\left[C+\ln r\right] & \text{if } x>0\\ 0 & \text{otherwise}\end{cases}.$$

See [Gradshteyn, p. 571]. Thus,

$$-\frac{e^{r\beta}}{\alpha}\int_0^{\infty}\ln(t+\beta)e^{-r(t+\beta)}dt=\frac{e^{r\beta}}{\alpha r}\left[C+\ln r\right].$$

$$k_0=\frac{1}{\alpha r}\ln\left(\frac{\alpha\beta}{\lambda}\right)+\frac{e^{r\beta}}{\alpha r}\left[C+\ln r\right]+\frac{1}{\alpha r},$$

$$\frac{1}{\alpha}\ln\left(\frac{\alpha\beta}{\lambda}\right)=rk_0-\frac{e^{r\beta}}{\alpha}\left[C+\ln r\right]-\frac{1}{\alpha}.$$

Finally, replace the expression that contains the Lagrange's multiplier into the consumption path

$$c_t=rk_0-\frac{1}{\alpha}\left[e^{r\beta}\left(C+\ln r\right)+1\right]+\frac{1}{\alpha}\left(\ln\left(\frac{1}{t+\beta}\right)+rt\right).$$

APPENDIX C. THE MODEL OF THE RISK AVERSION RATE AS RANDOM VARIABLE WITH INFINITE HORIZON

The maximizations problem is:

$$\text{Maximize} \quad \int_0^\infty -\mu e^{-\rho t}\left(\int_0^\infty e^{-\alpha(c_t+\mu)} d\alpha\right) dt$$

$$\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt} dt$$

So, first the inner integral is solved

$$\text{Maximize} \quad \int_0^\infty -\frac{\mu}{c_t+\mu} e^{-\rho t} dt$$

$$\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt} dt$$

$$\mathcal{L}(c_t,\lambda) = \frac{-\mu e^{-\rho t}}{(c_t+\mu)} + \lambda(k_0 - c_t)e^{-rt}$$

$$\mathcal{L}_{c_t} = \frac{\mu e^{-\rho t}}{(c_t+\mu)^2} - \lambda e^{-rt} = 0,$$

$$c_t = \left[\frac{\mu}{\lambda}\right]^{1/2} e^{\frac{r-\rho}{2}t} - \mu.$$

Next, replace this expression into the constraint and solve the integral

$$k_0 = \int_0^\infty \left(\left[\frac{\mu}{\lambda}\right]^{1/2} e^{\frac{r-\rho}{2}t} - \mu\right) e^{-rt} dt,$$

$$k_0 = \left(\frac{2}{r+\rho}\right)\left[\frac{\mu}{\lambda}\right]^{1/2} - \frac{\mu}{r}.$$

Then,

$$c_t = \left[k_0 + \frac{\mu}{r}\right]\left(\frac{r+\rho}{2}\right) e^{\frac{r-\rho}{2}t} - \mu\,.$$

APPENDIX D. CONVENTIONAL MODEL WITH FINTE HORIZON

Since the set up is the same as A.1, but for the time horizon, the Lagrangeans coincide. This means that

$$c_t = \frac{1}{\alpha}\ln\left(\frac{\alpha}{\lambda}\right) + \frac{1}{\alpha}(r-\rho)t.$$

However, the interval of the integral is from 0 to T.

$$k_0 = \int_0^T \left(\frac{1}{\alpha}\ln\left(\frac{\alpha}{\lambda}\right) + \frac{1}{\alpha}(r-\rho)t\right)e^{-rt}dt,$$

$$k_0 = -\frac{1}{\alpha r}\ln\left(\frac{\alpha}{\lambda}\right)\left[1-e^{-rT}\right] + \frac{(r-\rho)}{\alpha}\left[-\frac{T}{r}e^{-rT} + \frac{1}{r^2}\left[1-e^{-rT}\right]\right],$$

$$-\frac{1}{\alpha}\ln\left(\frac{\alpha}{\lambda}\right) = \left\{rk_0 - \frac{(r-\rho)}{\alpha}\left[\frac{1}{r}\left[1-e^{-rT}\right] - Te^{-rT}\right]\right\}\left[1-e^{-rT}\right]^{-1}.$$

Replacing this value into the consumption,

$$c_t = \left\{rk_0 - \frac{(r-\rho)}{\alpha}\left[\frac{1}{r}\left[1-e^{-rT}\right] - Te^{-rT}\right]\right\}\left[1-e^{-rT}\right]^{-1} + \frac{1}{\alpha}(r-\rho)t.$$

APPENDIX E. THE MODEL OF THE SUBJECTIVE DISCOUNT RATE AS RANDOM VARIABLE WITH FINITE HORIZON

As it was mentioned before Lagrangians generate the same result, so

$$c_t = \frac{1}{\alpha}\ln\left(\frac{\alpha\beta}{\lambda}\right) + \frac{1}{\alpha}rt + \frac{1}{\alpha}\ln\left(\frac{1}{t+\beta}\right).$$

Substituting c_t into the restriction,

$$k_0 = \int_0^T \left(\frac{1}{\alpha}\ln\left(\frac{\alpha\beta}{\lambda}\right) + \frac{1}{\alpha}rt + \frac{1}{\alpha}\ln\left(\frac{1}{t+\beta}\right)\right) e^{-rt}dt.$$

The first two terms of the integral are easy to calculate. Nevertheless, the last term of this integral represents a major challenge:

$$-\frac{1}{\alpha}\int_0^T \ln(t+\beta)e^{-rt}dt = -\frac{1}{\alpha}e^{r\beta}\int_0^T \ln(t+\beta)e^{-r(t+\beta)}dt = -\frac{1}{\alpha}e^{r\beta}(T+\beta)\int_0^1 \ln\left[y(T+\beta)\right]e^{-ry(T+\beta)}dy.$$

Obviously, there is a change of variable, $y = \frac{t+\beta}{T+\beta}$. It follows that

$$-\frac{1}{\alpha}e^{r\beta}(T+\beta)\int_0^1 \ln\left[y(T+\beta)\right]e^{-ry(T+\beta)}dy = -\frac{1}{\alpha}e^{r\beta}(T+\beta)\int_0^1 \left(\ln(T+\beta) + \ln y\right)e^{-ry(T+\beta)}dy,$$

where the first term is constant with respect to *y* and the second is solved according to [Gradshteyn, p. 571]. Then, the solution is

$$-\frac{1}{\alpha}e^{r\beta}(T+\beta)\int_0^1 \left(\ln(T+\beta) + \ln y\right)e^{-ry(T+\beta)}dy = \frac{1}{\alpha r}e^{r\beta}\ln(T+\beta)\left[e^{-r(T+\beta)} - 1\right]$$
$$-\frac{1}{\alpha r}e^{r\beta}\left[C + \ln r + \ln(T+\beta)\right].$$

Therefore,

$$k_0 = \frac{1}{\alpha r}\left[1 - e^{-rT} - rTe^{-rT}\right] + \frac{e^{r\beta}}{\alpha r}\left[C + \ln r + \ln(T+\beta)\right] + \frac{e^{r\beta}}{\alpha r}\ln(T+\beta)\left[e^{-r(T+\beta)} - 1\right]$$
$$+\frac{1}{\alpha r}\ln\left(\frac{\alpha\beta}{\lambda}\right)\left[1 - e^{-rT}\right],$$

$$\frac{1}{\alpha}\ln\left(\frac{\alpha\beta}{\lambda}\right) = \left\{r\alpha k_0 - \left[1 - e^{-rT}(1+rT)\right] - e^{r\beta}\left[C + \ln r + \ln(T+\beta)e^{-r(T+\beta)}\right]\right\}\left[\alpha\left(1 - e^{-rT}\right)\right]^{-1}.$$

There is just one step left, replace the expression that contains the Lagrange's multiplier into the consumption path

$$c_t = \frac{1}{\alpha} rt + \frac{1}{\alpha} \ln\left(\frac{1}{t+\beta} \right) + \left\{ r\alpha k_0 - \left[1 - e^{-rT} \left(1 + rT \right) \right] - e^{r\beta} \left[C + \ln r + \ln\left(T + \beta \right) e^{-r\left(T+\beta\right)} \right] \right\} \left[\alpha \left(1 - e^{-rT} \right) \right]^{-1}.$$

APPENDIX F. THE MODEL OF THE RISK AVERSION RATE AS RANDOM VARIABLE WITH FINITE HORIZON

Considering the maximization problem in this setting, we obtain that

$$c_t = \left[\frac{\mu}{\lambda}\right]^{1/2} e^{\frac{r-\rho}{2}t} - \mu.$$

Then, the allocation resource restriction is as follows

$$k_0 = \int_0^T \left(\left[\frac{\mu}{\lambda}\right]^{1/2} e^{\frac{r-\rho}{2}t} - \mu \right) e^{-rt} dt,$$

$$k_0 = \left(\frac{2}{r+\rho}\right)\left[1 - e^{-\frac{r-\rho}{2}T}\right]\left[\frac{\mu}{\lambda}\right]^{1/2} - \frac{\mu}{r}\left[1 - e^{-rT}\right],$$

$$\left[\frac{\mu}{\lambda}\right]^{1/2} = \left\{k_0 + \frac{\mu}{r}\left[1 - e^{-rT}\right]\right\}\left(\frac{2}{r+\rho}\right)\left[1 - e^{-\frac{r-\rho}{2}T}\right]^{-1}.$$

Therefore,

$$c_t = \left\{k_0 + \frac{\mu}{r}\left[1 - e^{-rT}\right]\right\}\left(\frac{r+\rho}{2}\right)\left[1 - e^{-\frac{r-\rho}{2}T}\right]^{-1} e^{\frac{r-\rho}{2}t} - \mu.$$

APPENDIX G. BIVARIATE MODEL (FTH)

Maximize $\int_0^\infty \left(\int_0^\infty \left(\int_0^T -e^{-\alpha c_t} e^{-\rho t} dt \right) \mu e^{-\mu\alpha} d\alpha \right) \lambda e^{-\lambda\rho} d\rho$

subject to $k_0 = \int_0^T c_t e^{-rt} dt$

Applying twice the Fubini's theorem to the objective function

Maximize $\int_0^T \int_0^\infty \int_0^\infty -\mu\lambda e^{-\alpha(c_t+\mu)} e^{-\rho(t+\lambda)} d\alpha d\rho d$

Devoloping the inner integrals of the objective function.

$$\int_0^T \int_0^\infty -\mu\lambda e^{-\rho(t+\lambda)} \int_0^\infty e^{-\alpha(c_t+\mu)} d\alpha d\rho dt = \int_0^T \int_0^\infty \frac{-\mu\lambda e^{-\rho(t+\lambda)}}{c_t+\mu} d\rho dt = \int_0^T \frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)} dt.$$

Hence, we have now a new expression for the maximization problem:

Maximize $\int_0^T \frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)} dt$

subject to $\int_0^T \left(rk_t - \dot{k}_t \right) e^{-rt} dt = \int_0^T c_t e^{-rt} dt$

The Lagrangian is set up in this way:

$$\mathfrak{L}(c_t, \lambda) = \frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)} + \beta\left(rk_t - \dot{k}_t - c_t \right) e^{-rt}$$

Differentiating with respect to c_t

$$\mathfrak{L}_{c_t} = \frac{\mu\lambda}{(t+\lambda)(c_t+\mu)^2} - \beta e^{-rt} = 0$$

Isolating c_t

$$c_t = \sqrt{\frac{\mu\lambda}{\beta} \frac{e^{rt}}{(t+\lambda)}} - \mu$$

Substituting this expression in the restriction

$$k_0 = \int_0^T \sqrt{\frac{\mu\lambda}{\beta} \frac{e^{rt}}{(t+\lambda)}} e^{-rt} dt - \int_0^T \mu e^{-rt} dt,$$

$$k_0 = \sqrt{\frac{\mu\lambda}{\beta}} \int_0^T \frac{1}{(t+\lambda)^{1/2}} e^{-\frac{r}{2}t} dt - \frac{\mu}{r}\left(1 - e^{-rT}\right),$$

$$\int_0^T \frac{1}{(t+\lambda)^{1/2}} e^{-\frac{r}{2}t} dt = 2\int_0^T \frac{1}{2(t+\lambda)^{1/2}} e^{-\frac{r}{2}t} dt$$

sea $z = (t+\lambda)^{1/2}, \quad dz = \dfrac{1}{2(t+\lambda)^{1/2}} dt$. Then,

changing once again the variable

$y = \sqrt{r} z, \quad dy = \sqrt{r} dz$

$$2e^{\frac{r\lambda}{2}} \frac{1}{\sqrt{r}} \int_{\sqrt{r\lambda}}^{\sqrt{r(T+\lambda)}} e^{-\frac{1}{2}y^2} dy = 2e^{\frac{r\lambda}{2}} \frac{\sqrt{2\pi}}{\sqrt{r}} \int_{\sqrt{r\lambda}}^{\sqrt{r(T+\lambda)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

$$= 2e^{\frac{r\lambda}{2}} \frac{\sqrt{2\pi}}{\sqrt{r}} \left[\Phi\left(\sqrt{r(T+\lambda)}\right) - \Phi\left(\sqrt{r\lambda}\right) \right].$$

Next,

$$k_0 = \sqrt{\frac{\mu\lambda}{\beta}} \left(2e^{\frac{r\lambda}{2}} \frac{\sqrt{2\pi}}{\sqrt{r}} \right) \left[\Phi\left(\sqrt{r(T+\lambda)}\right) - \Phi\left(\sqrt{r\lambda}\right) \right] - \frac{\mu}{r}\left(1 - e^{-rT}\right).$$

The only unknown parameter is the Lagrange multiplier, so

$$\sqrt{\frac{\mu\lambda}{\beta}} = \frac{(y+\mu)\left(1-e^{-rT}\right)}{\left[\sqrt{8\pi r} e^{\frac{r\lambda}{2}} \left[\Phi\left(\sqrt{r(T+\lambda)}\right) - \Phi\left(\sqrt{r\lambda}\right) \right] \right]}.$$

Therefore,

$$c_t = \frac{e^{\frac{r(t-\lambda)}{2}}}{\sqrt{t+\lambda}} \frac{\left(rk_0 + \mu\left(1-e^{-rT}\right)\right)}{\left[\sqrt{8\pi r} \left[\Phi\left(\sqrt{r(T+\lambda)}\right) - \Phi\left(\sqrt{r\lambda}\right) \right] \right]} - \mu.$$

APPENDIX H. CAPITAL PATH (FTH)

After substituting the optimal consumption of the average individual in the constraint, the capital path satisfies

$$k_t = k_0 e^{rt} - e^{rt}\int_0^t c_s e^{-rs}\mathrm{d}s$$

$$= k_0 e^{rt} - e^{rt}\int_0^t \left\{ \frac{e^{\frac{r}{2}(s-\lambda)}}{\sqrt{s+\lambda}} \left[\frac{\left(rk_0 + \mu\left(1-e^{-rT}\right)\right)}{\sqrt{8\pi r}\left(\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)} \right] - \mu \right\} e^{-rs}\mathrm{d}s.$$

$$= k_0 e^{rt} + e^{rt}\int_0^t \mu e^{-rs} ds - \frac{e^{rt}}{\sqrt{8\pi r}} \left[\frac{\left(rk_0 + \mu\left(1-e^{-rT}\right)\right)}{\left(\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)} \right] \int_0^t \frac{e^{-\frac{r}{2}(s+\lambda)}}{(s+\lambda)^{\frac{1}{2}}}\mathrm{d}s$$

$$= k_0 e^{rt} - e^{rt}\frac{\mu}{r}(e^{-rt}-1) - e^{rt}e^{-\frac{r\lambda}{2}} \left[\frac{\left(rk_0 + \mu\left(1-e^{-rT}\right)\right)}{\sqrt{8\pi r}\left(\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)} \right] \int_\lambda^{t+\lambda} \frac{e^{-\frac{rs}{2}}}{(s+\lambda)^{\frac{1}{2}}}\mathrm{d}s.$$

Now the following change of variable is used:

$$\omega = (s+\lambda)^{\frac{1}{2}},$$

Hence,

$$k_t = e^{rt}\left[k_0 + \frac{\mu}{r}\right] - \frac{\mu}{r} - 2e^{rt}e^{-\frac{r\lambda}{2}} \left[\frac{\left(rk_0 + \mu\left(1-e^{-rT}\right)\right)}{\sqrt{8\pi r}\left(\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)} \right] \int_{\sqrt{\lambda}}^{\sqrt{t+\lambda}} e^{-\frac{r}{2}\left(\omega^2-\lambda\right)}\mathrm{d}\omega$$

$$= e^{rt}\left[k_0 + \frac{\mu}{r}\right] - \frac{\mu}{r} - 2e^{rt} \left[\frac{\left(rk_0 + \mu\left(1-e^{-rT}\right)\right)}{\sqrt{8\pi r}\left(\Phi\left(\sqrt{r(T+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)} \right] \int_{\sqrt{\lambda}}^{\sqrt{t+\lambda}} e^{-\frac{r}{2}\omega^2}\mathrm{d}\omega.$$

Now, let

$$J = \int_{\sqrt{\lambda}}^{\sqrt{t+\lambda}} e^{-\frac{r}{2}\omega^2}\mathrm{d}\omega \quad \text{and} \quad u = \sqrt{r}\omega,$$

then

$$J = \frac{1}{\sqrt{r}}\int_{\sqrt{r\lambda}}^{\sqrt{r(t+\lambda)}} e^{-\frac{1}{2}u^2}du = \frac{\sqrt{2\pi}}{\sqrt{r}}\int_{\sqrt{r\lambda}}^{\sqrt{r(t+\lambda)}} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du = \frac{\sqrt{2\pi}}{\sqrt{r}}\left[\Phi\left(\sqrt{r(t+\lambda)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right].$$

Hence,

$$k_t = e^{rt}\left[k_0 + \frac{\mu}{r}\right] - \frac{\mu}{r} - 2e^{rt}\left[\frac{\left(rk_0 + \mu\left(1 - e^{-rT}\right)\right)}{\sqrt{8\pi r}\left(\Phi\left(\sqrt{r\left(T+\lambda\right)}\right) - \Phi\left(\sqrt{r\lambda}\right)\right)}\right]\frac{\sqrt{2\pi}}{\sqrt{r}}\left[\Phi\left(\sqrt{r\left(t+\lambda\right)}\right) - \Phi\left(\sqrt{r\lambda}\right)\right].$$

$$k_t = \left[k_0 + \frac{\mu}{r}\right]e^{rt} - \frac{\mu}{r} - \frac{e^{rt}}{r}\left[\frac{rk_0 + \mu\left(1 - e^{-rT}\right)}{\Phi\left(\sqrt{r\left(T+\lambda\right)}\right) - \Phi\left(\sqrt{r\lambda}\right)}\right]\left[\Phi\left(\sqrt{r\left(t+\lambda\right)}\right) - \Phi\left(\sqrt{r\lambda}\right)\right].$$

APPENDIX I. WELFARE FUNCTION (FTH)

If the integral appearing in the welfare function is denoted by

$$I = \int_0^T \frac{e^{-\frac{rs}{2}}}{(s+\lambda)^{\frac{1}{2}}}\text{ds}$$

and the following change of variable is used

$$\omega = (s+\lambda)^{\frac{1}{2}},\ \omega^2 - \lambda = s,\ \text{d}\omega = \frac{1}{2(s+\lambda)^{\frac{1}{2}}}\text{d}s,$$

then

$$\begin{aligned} I &= 2\int_{\sqrt{\lambda}}^{\sqrt{T+\lambda}} e^{-\frac{r}{2}\left(\omega^2-\lambda\right)}\text{d}\omega = 2e^{\frac{r\lambda}{2}}\int_{\sqrt{\lambda}}^{\sqrt{T+\lambda}} e^{-\frac{r}{2}\omega^2}\text{d}\omega \\ &= 2e^{\frac{r\lambda}{2}}\frac{\sqrt{2\pi}}{\sqrt{r}}\int_{\sqrt{\lambda}}^{\sqrt{T+\lambda}}\frac{\sqrt{r}}{\sqrt{2\pi}}e^{-\frac{r}{2}\omega^2}\text{d}\omega. \end{aligned}$$

Hence,

$$I = 2e^{\frac{r\lambda}{2}}\frac{\sqrt{2\pi}}{\sqrt{r}}\left[\Phi\left(\sqrt{T+\lambda}\right) - \Phi\left(\sqrt{\lambda}\right)\right],$$

thus

$$W = \frac{-8\pi(\mu\lambda)e^{r\lambda}}{rk_0 + \mu\left(1-e^{-rT}\right)}\left[\Phi\left(\sqrt{T+\lambda}\right) - \Phi\left(\sqrt{\lambda}\right)\right]\left[\Phi\left(\sqrt{r(T+\lambda)}\right) - \Phi\left(\sqrt{r\lambda}\right)\right].$$

APPENDIX J. DERIVATIVES OF THE WELFARE FUNCTION (FTH)

In this section the partial derivates of the welfare function are computed. Notice that

$$\frac{\partial W}{\partial k_0}=\frac{8\pi\mu\lambda r e^{r\lambda}}{\left[rk_0+\mu\left(1-e^{-rT}\right)\right]^2}\left\{\Phi\left(\sqrt{T+\lambda}\right)-\Phi\left(\sqrt{\lambda}\right)\right\}\left[\Phi\left(\sqrt{r\left(T+\lambda\right)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]>0.$$

$$\frac{\partial W}{\partial \mu}=\frac{-8\pi\lambda r k_0 e^{r\lambda}}{\left[rk_0+\mu\left(1-e^{-rT}\right)\right]^2}\left\{\Phi\left(\sqrt{T+\lambda}\right)-\Phi\left(\sqrt{\lambda}\right)\right\}\left[\Phi\left(\sqrt{r\left(T+\lambda\right)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]<0;$$

$$\begin{aligned}\frac{\partial W}{\partial T}=&\frac{8\pi\mu^2\lambda r e^{r\lambda}}{\left[rk_0+\mu\left(1-e^{-rT}\right)\right]^2}\left\{\Phi\left(\sqrt{T+\lambda}\right)-\Phi\left(\sqrt{\lambda}\right)\right\}\left[\Phi\left(\sqrt{r\left(T+\lambda\right)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]\\&-\left[\frac{8\pi\mu\lambda e^{r\lambda}}{rk_0+\mu\left(1-e^{-rT}\right)}\right]\left[\frac{1}{\sqrt{8\pi\left(T+\lambda\right)}}\right]\left[\left(\Phi\left(\sqrt{r\left(T+\lambda\right)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)e^{-\frac{1}{2}\left(T+\lambda\right)}\right.\\&\left.+\sqrt{r}\left\{\Phi\left(\sqrt{T+\lambda}\right)-\Phi\left(\sqrt{\lambda}\right)\right\}e^{-\frac{r}{2}\left(T+\lambda\right)}\right].\end{aligned}$$

If T is large enough, then the second term of the right-hand side of the above equation goes to zero exponentially. Thus, $\frac{\partial W}{\partial T}>0.$ Moreover,

$$\begin{aligned}\frac{\partial W}{\partial r}=&\frac{-8\pi\mu\lambda e^{r\lambda}\left\{\Phi\left(\sqrt{T+\lambda}\right)-\Phi\left(\sqrt{\lambda}\right)\right\}}{rk_0+\mu\left(1-e^{-rT}\right)}\left[\left[\Phi\left(\sqrt{r\left(T+\lambda\right)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]\left\{\lambda-\frac{k_0}{rk_0+\mu\left(1-e^{-rT}\right)}\right\}\right.\\&\left.+\frac{e^{-\frac{r\lambda}{2}}\left[\left(\sqrt{T+\lambda}\right)e^{-\frac{rT}{2}}-\sqrt{\lambda}\right]}{\sqrt{8\pi r}}\right].\end{aligned}$$

If $\left[\Phi\left(\sqrt{r\left(T+\lambda\right)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]\left\{\lambda-\frac{k_0}{rk_0+\mu\left(1-e^{-rT}\right)}\right\}+e^{-\frac{r\lambda}{2}}\frac{\left[\left(\sqrt{T+\lambda}\right)e^{-\frac{rT}{2}}-\sqrt{\lambda}\right]}{\sqrt{8\pi r}}>0,$

then, $\frac{\partial W}{\partial r} < 0$; otherwise, if

$$\left[\Phi\left(\left[r\left(T+\lambda\right)\right]^{1/2}\right)-\Phi\left(\left[r\lambda\right]^{1/2}\right)\right]\left\{\lambda-\frac{k_0}{rk_0+\mu\left(1-e^{-rT}\right)}\right\}+e^{-\frac{r\lambda}{2}}\frac{\left[\left(T+\lambda\right)^{1/2}e^{-\frac{rT}{2}}-\lambda^{1/2}\right]}{\left(8\pi r\right)^{1/2}}<0,$$

then, $\frac{\partial W}{\partial r} > 0$. Finally, notice that

$$\frac{\partial W}{\partial \lambda}=-\frac{8\pi\mu e^{r\lambda}}{rk_0+\mu\left(1-e^{-rT}\right)}\left\{\left\{\Phi\left(\sqrt{T+\lambda}\right)-\Phi\left(\sqrt{\lambda}\right)\right\}\left[\Phi\left(\sqrt{r\left(T+\lambda\right)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right]\left(1+r\lambda\right)\right.$$

$$\left.+\frac{\lambda}{\sqrt{8\pi}}\left[\frac{e^{-\frac{rT}{2}}}{\sqrt{T+\lambda}}-\frac{1}{\sqrt{\lambda}}\right]\left[e^{-\frac{\lambda}{2}}\left(\Phi\left(\sqrt{r\left(T+\lambda\right)}\right)-\Phi\left(\sqrt{r\lambda}\right)\right)+\frac{e^{-\frac{r\lambda}{2}}}{\sqrt{r}}\left(\Phi\left(\sqrt{T+\lambda}\right)-\Phi\left(\sqrt{\lambda}\right)\right)\right]\right\}<0.$$

APPENDIX K. BIVARIATE MODEL (ITH)

In this appendix the steps to obtain the results in the bivariate model of the infinite time horizon are shown:

$$\text{Maximize} \quad \int_0^\infty \left(\int_0^\infty \left(\int_0^\infty -e^{-\alpha c_t} e^{-\rho t} \mathrm{d}t \right) \mu e^{-\mu\alpha} \mathrm{d}\alpha \right) \lambda e^{-\lambda\rho} \mathrm{d}\rho$$

$$\text{subject to} \quad k_0 = \int_0^\infty c_t e^{-rt} \mathrm{d}t.$$

Fubini's theorem is applied twice to the objective,

$$\text{Maximize} \quad \int_0^\infty \int_0^\infty \int_0^\infty -\mu\lambda e^{-\alpha(c_t+\mu)} e^{-\rho(t+\lambda)} \mathrm{d}\alpha \mathrm{d}\rho \mathrm{d}t$$

$$\text{subject to} \qquad k_0 = \int_0^\infty c_t e^{-rt} \mathrm{d}t.$$

Solving the inner integrals,

$$\int_0^\infty \int_0^\infty -\mu\lambda e^{-\rho(t+\lambda)} \int_0^\infty e^{-\alpha(c_t+\mu)} \mathrm{d}\alpha \mathrm{d}\rho \mathrm{d}t = \int_0^\infty \int_0^\infty \frac{-\mu\lambda e^{-\rho(t+\lambda)}}{c_t+\mu} \mathrm{d}\rho \mathrm{d}t = \int_0^\infty \frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)} \mathrm{d}t.$$

Thus, the objective function has changed and the decision maker faces the next maximization problem:

$$\text{Maximize} \qquad \int_0^\infty \frac{-\mu\lambda}{(t+\lambda)(c_t+\mu)} \mathrm{d}t$$

$$\text{subject to} \qquad k_0 = \int_0^\infty c_t e^{-rt} \mathrm{d}t.$$

Later, a Lagrangian is formulated:

$$c_t = \sqrt{\frac{\mu\lambda}{\beta} \frac{e^{rt}}{(t+\lambda)}} - \mu = \sqrt{\frac{\mu\lambda}{\beta}} \sqrt{\frac{e^{rt}}{(t+\lambda)}} - \mu.$$

Sustituyendo esta expresión en la restricción

$$k_0 = \int_0^\infty \sqrt{\frac{\mu\lambda}{\beta} \frac{e^{rt}}{(t+\lambda)}} e^{-rt} \mathrm{d}t - \int_0^\infty \mu e^{-rt} \mathrm{d}t = \sqrt{\frac{\mu\lambda}{\beta}} \int_0^\infty \frac{1}{(t+\lambda)^{1/2}} e^{-\frac{r}{2}t} \mathrm{d}t - \frac{\mu}{r},$$

$$\int_0^\infty \frac{1}{(t+\lambda)^{1/2}} e^{-\frac{r}{2}t} \mathrm{d}t = 2\int_0^\infty \frac{1}{2(t+\lambda)^{1/2}} e^{-\frac{r}{2}t} \mathrm{d}t.$$

$$\text{Let } z = (t+\lambda)^{1/2}, \quad \mathrm{d}z = \frac{1}{2(t+\lambda)^{1/2}} \mathrm{d}t. \text{ Then,}$$

$$2\int_{\sqrt{\lambda}}^{\infty} e^{-\frac{r}{2}\left(z^2-\lambda\right)}\mathrm{d}z = 2e^{\frac{r\lambda}{2}}\int_{\sqrt{\lambda}}^{\infty} e^{-\frac{1}{2}\left(\frac{z}{\left(1/\sqrt{r}\right)}\right)^2}\mathrm{d}z = 2e^{\frac{r\lambda}{2}}\frac{1}{\sqrt{r}}\int_{\sqrt{\lambda}}^{\infty} e^{-\frac{1}{2}\left(\frac{z}{\left(1/\sqrt{r}\right)}\right)^2}\sqrt{r}\mathrm{d}z.$$

With a new change of variable

$$y = \sqrt{r}z, \quad \mathrm{d}y = \sqrt{r}\mathrm{d}z$$

$$2e^{\frac{r\lambda}{2}}\frac{1}{\sqrt{r}}\int_{\sqrt{r\lambda}}^{\infty} e^{-\frac{1}{2}y^2}\mathrm{d}y = 2e^{\frac{r\lambda}{2}}\frac{\sqrt{2\pi}}{\sqrt{r}}\int_{\sqrt{r\lambda}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\mathrm{d}y = 2e^{\frac{r\lambda}{2}}\frac{\sqrt{2\pi}}{\sqrt{r}}\left[1-\Phi\left(\sqrt{r\lambda}\right)\right].$$

Then,

$$k_0 = \sqrt{\frac{\mu\lambda}{\beta}}\left[2e^{\frac{r\lambda}{2}}\frac{\sqrt{2\pi}}{\sqrt{r}}\left[1-\Phi\left(\sqrt{r\lambda}\right)\right]\right] - \frac{\mu}{r}.$$

As usual, only β is unknown, so

$$\sqrt{\frac{\mu\lambda}{\beta}} = \frac{rk_0 + \mu}{\left[\sqrt{8\pi r}e^{\frac{r\lambda}{2}}\left[1-\Phi\left(\sqrt{r\lambda}\right)\right]\right]}.$$

Therefore,

$$c_t = \frac{e^{\frac{rt}{2}}}{\sqrt{t+\lambda}}\frac{rk_0 + \mu}{\left[\sqrt{8\pi r}e^{\frac{r\lambda}{2}}\left[1-\Phi\left(\sqrt{r\lambda}\right)\right]\right]} - \mu.$$

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