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Probabilistic Properties of Solutions of some Interest Rate Models

A Doctoral Dissertation

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by

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Venga tu Reino!

Vince in Bono Malum

I dedicate this doctoral dissertation with love …

to my son Eric Alessandro,

to my parents, Elia and Melquiades, and

to my brothers and sisters

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S.F.E.

Abstract

The thesis investigates mainly probabilistic properties of solutions of many standard interest rate models given by stochastic differential equations. Precisely it studies solutions of such models by using the standard Itô's formula and also provides an alternative elementary method other than this formula. Moreover, it obtains properties of these solutions like expectation, variance, p^{th}-moments, $p \geq 2$, and some concepts of stochastic stability. Furthermore, the thesis also obtains generalizations of the above theory by considering some of these interest rate models with Poisson jumps.

Resumen

La tesis investiga principalmente propiedades probabilísticas de soluciones de varios modelos de tasa de interés descritos por ecuaciones diferenciales estocásticas. Con precisión, la tesis estudia soluciones de tales modelos usando la fórmula estándar de Itô y también provee un método sencillo alternativo a dicha fórmula. Además la tesis obtiene propiedades de estas soluciones como esperanza, varianza, p-ésimo momentos, $p \geq 2$, y algunos conceptos de estabilidad. Adicionalmente, la tesis obtiene generalizaciones de la teoría descrita arriba considerando algunos modelos de tasa de interés con saltos de Poisson.

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Chapter 1

Introduction

A fundamental element in economics is the concept of the interest rate as the price of money. The history of the interest rate can be traced back to the history of the money itself. From the parable of talents, Matthew 25:14-30, the interest has been considered as a source of making more money and the process was considered a magical one, see Salinas, 2011. A level of interest rate can have a profound impact on inflation and employment, and therefore the welfare of the people. It can also have an important effect in financial markets such as stock and bond markets. Changes in interest rates can become a major source of risk for banks, investment firms, insurance companies, and multinational corporations, among other investors. For these players in financial markets, modeling the term-structure movements of interest rates is of fundamental importance and a challenging task.

Central banks are of paramount importance to the economy, and financial markets in particular, because these institutions change the money supply through open-market operations in bond markets, and further set a certain level of interest rates through monetary policy. For financial markets (stock or bond markets) a level of the interest rate can induce a bull or a bear market, meaning important gains or losses for financial institutions, investment firms or the public at large, when trading financial instruments.

Derivative securities such as stock options, options on stock indices and currencies, exotic options, futures, swaps, fixed income securities, credit derivatives, weather, energy, and insurance derivatives, interest rate derivatives, have become of central importance for modern corporate finance and investment. The trading of financial assets in the economy has increased dramatically in modern times with the availability of a broader availability of classic (securities) and more sophisticated financial instruments (i.e., derivative securities, including fixed income). This trend has generated a lot of literature in the areas of financial mathematics and financial economics, see Jeanblanc, Yor and Chesney, 2009, and Duffie 2001, among others.

Let us consider the so called fixed income markets, which include financial instruments such as government bonds traded in government debt markets, zero coupon bonds, floating rate coupon bonds, and municipal bonds. When modeling the price of a bond, it can be described in different ways depending on the treatment given to the interest rate process involved. A simple example of a bond is a bank account carrying a constant interest rate, $r > 0$. According to this model the price of a bond evolves as follows

$$\begin{aligned} \frac{dB(t)}{dt} &= rB(t), \quad t > 0, \\ B(0) &= B_0. \end{aligned} \tag{1.1}$$

in this case the price of the bond is given by $B(t) = B_0 e^{rt}$. An implicit assumption is that the time interval involved is sufficiently small.

An interest rate process can be modeled as a deterministic function of time, i.e., $r = r(t)$, where $t \geq 0$, and $B(0) = B_0$. Then the price of a bond is obtained as

$$B(t) = B_0 \exp\left\{\int_0^t r(s)ds\right\}, \quad t \geq 0. \tag{1.2}$$

We refer to Svishchuk and Kalemanova, 2000, and Govindan and Acosta, 2008, for details.

Black and Scholes, 1973, and Merton, 1973, proposed a model for the valuation of options as a stochastic process, that is, the underlying asset price $S(t)$ follows a geometric Brownian motion given by the following Itô SDE :

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW(t), \quad t > 0, \qquad (1.3) \\ S(0) &= S_0, \end{aligned}$$

where μ is a real constant and σ is positive constant. Equation (1.3) is known as the Black-Scholes-Merton model. In 1997, Scholes and Merton received the Nobel prize in economics for their work while Black passed away in 1995.

Interest rates can vary due to a variety of factors which are of a random nature. The seminal stochastic interest rate model in the field was proposed by Vasicek in 1977. Vasicek assumes a stochastic interest rate process as underlying the bond valuation process where the interest rate $r(t)$ follows an Ornstein-Uhlenbeck process as described by the following SDE:

$$\begin{aligned} dr(t) &= [\alpha - \beta r(t)]dt + \sigma dW(t), \quad t > 0, \qquad (1.4) \\ r(0) &= r_0, \end{aligned}$$

where α, β and σ are positive constants. Equation (1.4) is known as the Vasicek model for interest rates. Notice that the Vasicek interest rate model may allow the possibility of having negative interest rates.

We are interested in models of the interest rate in continuous time given by Itô stochastic differential equations (SDEs) studied by the rules of Itô stochastic calculus, see Cairns, 2004, Filipović, 2009, Shreve, 2004, Gibson et al, 2010, Brigo and Mercurio, 2007, Svoboda, 2004, and Veronesi, 2011.

A model that precludes the possibility of negative interest rates is the Cox-Ingersoll-Ross (CIR) interest rate model proposed by Cox, Ingersoll, and Ross in 1985b. The CIR model for the interest rate process $r(t)$ is given by the SDE:

$$\begin{aligned} dr(t) &= [\alpha - \beta r(t)]dt + \sigma\sqrt{r(t)}dW(t), \quad t > 0, \\ r(0) &= r_0, \end{aligned} \tag{1.5}$$

where α, β and σ are positive constants.

Hull and White, 1990, propose the following generalized interest rate model

$$\begin{aligned} dr(t) &= [\alpha(t) - \beta(t)r(t)]dt + \sigma(t)r^b(t)dW(t), \quad t > 0, \\ r(0) &= r_0, \end{aligned} \tag{1.6}$$

which still may allow negative interest rates, where $\alpha(t)$, $\beta(t)$ and $\sigma(t)$ are deterministic functions and b is a positive number. When α, β and σ are positive constants and $b = 0$, the Vasicek interest rate model (1.4) is obtained, as a special case of (1.6). If $b = 1/2$, (1.6) reduces to the CIR interest rate model (1.5).

In equation (1.6), when β and σ are positive constants and $b = 0$, the following well known form of the Hull-White interest rate model is obtained:

$$\begin{aligned} dr(t) &= [\alpha(t) - \beta r(t)]dt + \sigma dW(t), \quad t > 0, \\ r(0) &= r_0. \end{aligned} \tag{1.7}$$

The thesis considers also other models such as Extended Hull-White, Hull-White: A CIR style extension, Brennan Schwartz, Exponential Vasicek, Dothan, Mercurio-Moraleda, Black-Derman-Toy, and Black-Karasinki, among others.

Our research interests also include the study of continuous-time interest rate models with discontinuities. A motivation for this type of research comes from a seminal paper published by R. C. Merton in 1976 in the context of option pricing when underlying stock returns are discontinuous. Merton considers that the total change in the underlying stock price of an option could be as a result of "normal" vibrations, when business day news or events arise that move the market, and "abnormal" vibrations show up when non-expected and perhaps worrisome news hit the market. This last component can be modeled by a pure jump process, see Merton 1976.

In the context of continuous-time interest rate models we consider jump processes such as the compound Poisson jump process (Lévy processes) and the compensated compound Poisson jump process described in Merton 1976, Applebaum, 2009, and Bertoin, 1996.

1.1 Objectives of the Thesis

The objectives of this doctoral thesis are

1. to study an alternative to Itô's formula to find the solution process of some standard interest rate models such as Vasicek, Cox-Ingersoll-Ross (CIR), Hull-White (including extended Hull-White, and a CIR style extension of the Hull-White), Brennan-Schwartz, exponential Vasicek, Dothan, extended exponential Vasicek, Mercurio-Moraleda, a generalized interest rate model and the Brownian bridge,

2. to obtain the solution process by using Itô's product rule to some interest rate models with Poisson jumps (Lévy processes) such as Black-Derman-Toy and Black-Karasinski,

3. to obtain moment properties of some interest rate models such as Vasicek, Brennan-Schwartz, Hull-White (including an extension of Hull-White), Black-Derman-Toy, Black-Karasinski and Dothan and also some interest rate models with Poisson jumps such as Vasicek, Black-Derman-Toy and Black-Karasinski,

4. to obtain stability properties like boundedness in probability uniformly in t of Brennan-Schwartz model, exponential mean square stability of Brennan-Schwartz, Hull-White and Dothan models, and asymptotic quadratic mean of CIR with Poisson jumps.

1.2 Format of the Thesis

The format of this thesis is as follows:

In Chapter 2, we give preliminaries beginning with the basic concepts from the theory of probability and stochastic processes, then Itô's stochastic calculus, stochastic differential equations (SDEs), existence and uniqueness of a solution of SDEs. Next, some fundamentals such as Poisson jump processes, Lévy processes and Itô's product rule are provided. The chapter concludes with the necessary definitions of stability that we are interested in this thesis.

Chapter 3 provides an alternative to Itô's formula to solve many standard interest rate models. In other words, it is shown that the elementary linear ordinary differential equation method extended to the stochastic equations provides solutions to stochastic interest rate models like Vasicek, Cox-Ingersoll-Ross, Hull-White, Extended Hull-White, Hull-White: A CIR style extension, Brennan-Schwartz, Exponential Vasicek, Dothan, Black-Derman-Toy, Black-Karasinski, Extended Exponential Vasicek, Mercurio-Moraleda, a generalized interest rate model, and the Brownian bridge. However, Itô's formula is also used to obtain solutions. It is interesting that the solutions obtained by both the methods coincide.

In Chapter 4, we obtain probabilistic properties of interest rate models including such models with jumps. To be precise, moment properties such as expectation, variance, and also p-moments, $p \geq 2$, are obtained for the interest rate models like Vasicek, Brennan-Schwartz, Hull-White, Black-Derman-Toy, Black-Karasinski, and Dothan. Such moment properties are also obtained for interest rate models with Poisson jumps like Vasicek, Cox-Ingersoll-Ross, Black - Derman -Toy, Black-Karasinki, and Dothan. Subsequently, stability properties like boundedness in probability uniformly in t, and exponential mean square stability are established for the models like Vasicek, and Brennan-Schwartz with Poisson jumps.

Chapter 2

Preliminaries

In this Section we give the basic theory from Applebaum, 2009, Arnold, 1974, Allen, 2007, Mikosch, 1999, Gard, 1988, Gihman and Skorohod, 1972, Gut, 2013, Khasminskii, 2012, Itô, 1951, Mood, Graybill and Boes, 1974, Oksensal, 2003, and Shreve, 2004.

2.1 Probability and Stochastic Processes

Let Ω be a nonempty space, and let $\mathcal{F}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is a σ-algebra provided that (i) the empty set $\emptyset$ belongs to $\mathcal{F}$, (ii) whenever a set A belongs to $\mathcal{F}$, its complement A^c also belongs to $\mathcal{F}$, and (iii) whenever a sequence of sets A_1, A_2, ... belongs to $\mathcal{F}$, the union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{F}$. The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition 2.1 [Shreve, 2004, p. 2] Let Ω be a nonempty set, and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A probability measure P is a function defined as $P(\Omega, \mathcal{F}) \to [0,1]$ such that

(i) $P(A) \geq 0$, for every $A \in \mathcal{F}$,

(ii) $P(\Omega) = 1$, and

(iii) whenever A_1, A_2, ... is a sequence of disjoint sets in $\mathcal{F}$, i.e., $A_i \cap Aj = \emptyset$, for $i \neq j$, then

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} P(A_n).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Definition 2.2 [Arnold,1974, p. 2] Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ denote measurable spaces. A mapping $X : \Omega \to \Omega'$ that assigns to every $\omega \in \Omega$ a member $\omega' = X(\omega)$ of Ω' is said to be $(\mathcal{F}, \mathcal{F}')$-measurable (and is called an Ω'-valued random variable on $(\Omega, \mathcal{F})$) if the preimages of measurable sets in Ω' are measurable sets in Ω, that is, for $A' \in \mathcal{F}'$,

$$\{\omega : X(\omega) \in A'\} = [X(\omega) \in A'] = X^{-1}(A') \in \mathcal{F}.$$

The set $\mathcal{F}(X)$ of preimages of measurable sets is itself a σ-algebra in Ω and is the smallest σ-algebra with respect to which X is measurable. It is called the σ-algebra generated by X in Ω.

Definition 2.3 [Arnold, 1974, p. 8] A random variable X is said to be P-integrable if the integral $\int X dP$ is finite.

(i) In probability theory, this integral is also called the expectation of the random variable X and is written as

$$E[X] = EX = \int_{\Omega} X dP. \tag{2.1}$$

(ii) The variance of X defined as

$$Var[X] = E[X - EX]^2 = EX^2 - [EX]^2. \tag{2.2}$$

Let $R = (-\infty, \infty)$ be the real line.

Definition 2.4 [Arnold, 1974, p. 13] Let X and X_n, where $n \geq 1$, denote R-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.

(i) If there exists a set of measure zero $\mathcal{N} \in \mathcal{F}$ such that, for all $\omega \notin \mathcal{N}$, the sequence of the $X_n(\omega) \in R$ converges in the usual sense to $X(\omega) \in R$, then $\{X_n\}$ is said to converge almost surely (a.s.) or with probability 1 (w.p. 1) to X. We write

$$\mathrm{a.s.} - \lim_{\mathrm{n}\to\infty} \mathrm{X_n} = \mathrm{X}.$$

(ii) If, for every $\epsilon > 0$,

$$P\{\omega : |X_n(\omega) - X(\omega)| > \epsilon\} \to 0, \quad n \to \infty,$$

then $\{X_n\}$ is said to converge stochastically or in probability to X, and we write

$$P - \lim_{n\to\infty} X_n = X.$$

Definition 2.5 [Arnold, 1974, p. 16] Two random variables X and Y are said to be independent if and only if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ where $f_{X,Y}(x,y)$ is their joint density function of random variables X and Y and $f_X(x)$ and $f_Y(y)$ are their marginal density functions. This can be extended to more than two random variables in an analogous manner.

Definition 2.6 Two functions defined on the probability space $(\Omega, \mathcal{F}, P)$ f and g are equal *a.s.* with respect to P, if $f(x) = g(x)$ when $x \notin \mathcal{N}$, for some $\mathcal{N} \in \mathcal{F}$ such that $P(\mathcal{N}) = 0$.

Theorem 2.1 [Arnold, 1974, p. 12] (Radon-Nikodym) Let ν and μ denote two measures defined on $(\Omega, \mathcal{F})$, and suppose that μ is sigma-finite. Then, ν is μ-continuous if and only if ν has a μ-density. This density is uniquely defined $[\nu]$. We then have

$$\int_\Omega X d\nu = \int_\Omega X \frac{d\nu}{d\mu} d\mu.$$

as long as one side of this equation is meaningful.

Definition 2.7 [Arnold, 1974, p. 18] (Conditional expectation) Let $X \in L^1(\Omega, \mathcal{F}, P)$ denote an R-valued random variable and let $\mathcal{G} \subset \mathcal{F}$ denote a sub-σ-algebra of $\mathcal{F}$. The probability space $(\Omega, \mathcal{G}, P)$ is a coarsening of the original one and X is in general, no longer $\mathcal{G}$-measurable. We seek now a $\mathcal{G}$-measurable coarsening Y of X that assumes, on the average, the same values as X, that is, an integrable random variable Y such that

$$\begin{aligned} &Y \text{is } \mathcal{G} - \text{measurable}, \\ &\int_C Y dP = \int_C X dP \quad \text{for all C} \in \mathcal{G}. \end{aligned}$$

According to the Radon-Nikodym theorem, there exists exactly one such Y, a.s. unique. It is called the conditional expectation of X under the condition $\mathcal{G}$. We write

$$Y = E[X|\mathcal{G}]. \tag{2.3}$$

Therefore, the conditional expectation is, for fixed X and $\mathcal{G}$, a function of $\omega \in \Omega$. It follows from the definition that, in particular

$$\begin{aligned} &E[E[X|\mathcal{G}]] = E[X] \quad \text{and} \\ &|E[X|\mathcal{G}]| = E[|X||\mathcal{G}] \quad \text{a.s..} \end{aligned}$$

See [Arnold, 1974, p.19] for other important properties of the conditional expectation.

The conditional probability $P(A|\mathcal{G})$ of an event A under the condition $\mathcal{G} \in \mathcal{F}$ is defined by

$$P(A|\mathcal{G}) = E[I_A|\mathcal{G}], \tag{2.4}$$

where I_A is the indicator function defined by

$$I_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Theorem 2.2 [Gut, 2013, p. 120] (Markov's inequality) Suppose that $E|X|^p < \infty$ for some $c > 0$, and let $X > 0$. Then,

$$P\Big\{|X| \geq c\Big\} \leq \frac{1}{c^p} E|X|^p.$$

Definition 2.8 [Mood et al,1974, p. 107] (Normal distribution) A random variable X is defined to be normally distributed if its probability density function is given by

$$f_X(x) = f_X(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\Big\{-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big\}, \quad -\infty < x < \infty,$$

where the parameters μ and σ satisfy $-\infty < \mu < \infty$ and $\sigma > 0$. These parameters μ and σ^2 turn out to be the mean and the variance of the distribution.

Definition 2.9 [Doob, 1953, p. 46] A collection of random variables with respect to a parameter, with respect to time in this case, denoted by $\{X(t), t \in T\}$, where $T = [0,\infty)$ is called a stochastic process. For simplicity, we write $\{X(t), t \geq 0\}$.

Definition 2.10 [Arnold, 1974, p. 25] A stochastic process $\{X(t), t \in T\}$ is called a Gaussian process if every finite linear combination of random variables $\{X(t), t \in T\}$ is normally distributed.

Definition 2.11 [Shreve, 2004, p. 51] Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0,T]$ there is a σ-algebra. Assume further that if $s \leq t$, then every set in $\mathcal{F}_s$ is also in $\mathcal{F}_t$. Then we call the collection of σ-algebras $\mathcal{F}_t$, $0 \leq t \leq T$ a filtration.

2.1.1 Markov Process

Let $\{X(t), t \in T\}$ be a stochastic process adapted to a filtration $\{\mathcal{F}_t, t \in T\}$, with parameter space $T \subset [0,\infty)$ and state space R. Then $\{X(t), t \in T\}$ is called a Markov process if it satisfies

$$P[X(t) \in B|\mathcal{F}_s] = P[X(t) \in B|X(s)], \quad \text{a.s.},$$

for $0 \leq s \leq t$ and for every $B \in \mathcal{B}$, see Arnold, 1974, p. 28.

2.1.2 White Noise

Definition 2.12 [Arnold, 1974, p. 50] A Gaussian white noise is a generalized stationary Gaussian stochastic process $\xi(t)$, for $-\infty < t < \infty$, with mean $E\xi(t) = 0$ and a constant spectral density

$f(\lambda)$ on the entire real axis. If $E\xi(s)\xi(t+s) = C(t)$ is the covariance function of $\xi(t)$, then

$$f(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}C(t)dt = \frac{C}{2\pi} \quad \text{for all } \lambda \in \mathrm{R},$$

where C is a positive constant.

2.1.3 Wiener Process

Definition 2.13 [Arnold, 1974, p. 46] A stochastic process $\{W(t), t \geq 0\}$ is called a Wiener process or a Brownian motion if it satisfies the following conditions: $W(0) = 0$, the random variable $W(t) - W(s)$ is normally distributed with $E[W(t) - W(s)] = 0$ and $Var[W(t) - W(s)] = t - s$, $W(t)$ has independent increments, i.e., for every $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, the random variables $W(t_1), W(t_2) - W(t_1), \ldots W(t_n) - W(t_{n-1})$ are independent of each other and the sample paths of $W(t)$ are continuous functions a.s..

Theorem 2.3 [Arnold, 1974, p. 48] Almost all sample functions of the Wiener process $W(t)$ are continuous but nowhere differentiable functions.

Proposition 2.1 [Arnold, 1974, p. 45] The Wiener process, $\{W(t), t \geq 0\}$, is a Markov process.

Proposition 2.2 [Arnold, 1974, p. 53] The white noise is the derivative of the Wiener process $\{W(t), t \geq 0\}$ when both processes are considered as generalized stochastic processes, that is

$$\xi(t) = \frac{dW(t)}{dt} = \dot{W}(t), \text{ or conversely} \tag{2.5}$$

$$W(t) = \int_0^t \xi(s)ds$$

in the sense of coincidence of the covariance functionals. See Arnold, 1974, p. 53, for details.

2.1.4 Martingale

Definition 2.14 [Arnold, 1974, p. 25] Let $(\Omega, \mathcal{F}, P)$ denote a probability space. Let $\{X(t), t \in [t_0, T]\}$ denote an R-valued stochastic process defined on $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}_t\}_{t\in[t_0,T]}$ denote an increasing family of sub-σ-algebras of $\mathcal{F}$, that is, one having the property

$$\mathcal{F}_s \subset \mathcal{F}_t \quad \text{for} \quad \mathrm{t_0 \le s \le t \le T}.$$

If $X(t)$ is $\mathcal{F}_t$-measurable and integrable for all t, then the pair $\{X(t), \mathcal{F}_t\}_{t\in[t_0,T]}$ is called a martingale if

$$E[X(t)|\mathcal{F}_s] = X(s), \quad \text{a.s.}, \tag{2.6}$$

for all s and t in $[t_0, T]$, when $s \le t$. If $X(t)$ is a real-valued process and if we replace the equality sign in the last expression with $\le$ or $\ge$, what we have is a supermartingale or a submartingale.

2.2 Stochastic Calculus

In this Section, we introduce Itô stochastic calculus, see Arnold, 1974, Itô, 1951, Gard, 1988, Gihman and Skorohod, 1972, Oksendal, 2003, and Shreve, 2004.

Definition 2.15 [Arnold, 1974, p. 63] Let t_0 denote a fixed nonnegative number. A family $\mathcal{F}_t$, for $t \ge t_0$, of sub-σ algebras of $\mathcal{F}$ is said to be nonanticipating with respect to the Wiener process $\{W(t), t \ge 0\}$ if it has the following properties:

(i) $\mathcal{F}_s \subset \mathcal{F}_t \quad (t_0 \le s \le t)$,

(ii) $\mathcal{F}_t \supset \mathcal{B}[t_0, t] \quad (t \ge t_0)$,

(iii) $\mathcal{F}_t$ is independent of $\mathcal{B}_t^+ \quad (t \ge t_0)$.

Note that $\mathcal{B}[t_0, t]$ is the σ-algebra generated by the Wiener process $\{W(t), t \ge t_0\}$ given by $\mathcal{B}[t_0, t] = \mathcal{U}(W(u) : t_0 \le u \le t)$. Since $\mathcal{B}_0^+ = \mathcal{B}[0, \infty)$ (aside from sets of measure 0), condition (iii) means, for instance, for $t = 0$, that $\mathcal{F}_0$ can contain only events that are independent of the entire Wiener process $\{W(t), t \geq 0\}$.

Definition 2.16 [Arnold,1974, p. 63] An R-valued function $G = G(s,\omega)$ defined on $[t_0, t] \times \Omega$ and measurable in (s,ω) is said to be nonanticipating (with respect to a family $\mathcal{F}_s$ of nonanticipating σ-algebras) if $G(s,\cdot)$ is $\mathcal{F}_s$-measurable for all $s \in [t_0, t]$. We denote by $M_2[t_0, t]$ the set of those nonanticipating functions defined on $[t_0, t] \times \Omega$ for which the sample functions $G(\cdot,\omega)$ are w.p. 1 in $L_2[t_0, t]$, that is,

$$\int_{t_0}^{t} |G(s,\omega)|^2 ds < \infty.$$

Here, the last integral is to be interpreted as the Lebesgue integral (which, for example, coincides with the Riemann integral in the case of continuous functions).

2.2.1 Itô's Stochastic Integral

The purpose of this section is to define Itô's stochastic integral

$$\int_{t_0}^{t} G dW = \int_{t_0}^{t} G(s) dW(s) = \int_{t_0}^{t} G(s,\omega) dW(s,\omega)$$

for arbitrary $t \geq t_0$ and all $G \in M_2[t_0, t]$, in two steps. In the first step, Itô's stochastic integral is defined for step functions in $M_2[t_0, t]$. In the next step, its definition is extended to the entire set $M_2[t_0, t]$ by means of an approximation of an arbitrary function with the aid of step functions, see Section 4.4 from Arnold, 1974.

Step 1 A function $G \in M_2[t_0, t]$ is called a step function if there exists a decomposition $t_0 < t_1 < \ldots < t_n = t$ such that $G(s) = G(t_{i-1})$ (note that we omit the variable ω) for all $s \in [t_{i-1}, t_i)$, where $i = 1, \ldots, n$. For such step functions the stochastic integral of G with respect to $W(t)$ is defined as the R-valued random variable

$$\int_{t_0}^{t} G dW = \int_{t_0}^{t} G(s) dW(s) = \sum_{i=1}^{n} G(t_{i-1})(W_{t_i} - W_{t_{i-1}}). \tag{2.7}$$

Step 2 The definition of the Itô stochastic integral for arbitrary function in $M_2[t_0, t]$ is given as follows:

We need the following lemma.

Lemma 2.1 [Arnold,1974, p. 69] Suppose that $G \in M_2[t_0, t]$ and that $G_n \in M_2[t_0, t]$ is a sequence of step functions for which G_n converges stochastically to G in the following sense:

$$P - \lim_{n\to\infty} \int_{t_0}^{t} |G(s) - G_n(s)|^2 ds = 0.$$

If we define

$$\int_{t_0}^{t} G_n(s) dW(s)$$

by (2.7), then G_n converges stochastically to $I(G)$,

$$P - \lim_{n\to\infty} \int_{t_0}^{t} G_n(s) dW(s) = I(G),$$

where $I(G)$ is a random variable that does not depend on the special choice of the sequence G_n.

As a consequence of the previous two steps, the following definition can be established.

Definition 2.17 [Arnold, 1974, p. 71] For every R-valued function $G \in M_2[t_0, t]$, the stochastic integral (or Itô's integral) of G with respect to an R- valued Wiener process $W(t)$ over the interval $[t_0, t]$ is defined as the random variable $I(G)$, which is a. s. uniquely determined in accordance with Lemma 2.1:

$$\int_{t_0}^{t} G dW = \int_{t_0}^{t} G(s) dW(s) = P - \lim_{n\to\infty} \int_{t_0}^{t} G_n dW, \tag{2.8}$$

where $\{G_n\}$ is a sequence of step functions in $M_2[t_0, t]$ that converges stochastically to G in the sense of

$$P-\lim_{n\to\infty}\int_{t_0}^{t}|G(s)-G_n(s)|^2ds=0.$$

Some properties of Itô's stochastic integral are summarized as follows:

Theorem 2.4 [Shreve, 2004, p. 134] Let T be a positive constant and let $G(t)$, $G_1(t)$, $G_2(t)$, $0 \le s \le t \le T$, be adapted stochastic processes that satisfy $E\int_0^T G^2(t)dt < \infty$. Then $I(t)=\int_0^t G(s)dW(s)$ defined by (2.8) has the following properties:

(i) (Continuity) As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.

(ii) (Adaptivity) For each t, $I(t)$ is $\mathcal{F}_t$-measurable.

(iii) (Linearity)

$$\int_0^t [aG_1(s)+bG_2(s)]dW(s)=a\int_0^t G_1(s)dW(s)+b\int_0^t G_2(s)dW(s), \quad \text{a, b} \in \text{R}.$$

(iv) (Martingale) $I(t)$ is a martingale.

(v) (Itô Isometry)

$$E\left|\int_0^t G(s)dW(s)\right|^2=E\int_0^t |G(s)|^2ds.$$

(vi) (Quadratic Variation) $[I,I](t)=\int_0^t G^2(s)ds$.

Theorem 2.5 [Shreve, 2004, p. 169] (Itô integral of a deterministic integrand) Let $W(s)$, $s \ge 0$, be a Wiener process, and let $G(s)$ be a nonrandom function of time. Define $I(t)=\int_0^t G(s)dW(s)$. For each $t \ge 0$, the random variable $I(t)$ is normally distributed with

$$E[I(t)]=E\int_0^t G(s)dW(s)=0, \tag{2.9}$$

and

$$Var[I(t)] = Var\Big[\int_0^t G(s)dW(s)\Big] = \int_0^t G^2(s)ds. \tag{2.10}$$

2.2.2 Itô's Formula

We now give Itô's formula which will be crucial in this thesis.

Theorem 2.6 [Oksendal, 2003, p. 44] (Itô's Formula)
Let $X(t)$ be an Itô process of the form

$$dX(t) = f(t, X(t))dt + G(t, X(t))dW(t), \quad t > 0.$$

Let $g(t,x) \in C^2([0,\infty) \times R)$ (i.e. g is twice continuously differentiable on $[0,\infty) \times R)$. Then

$$Y(t) = g(t, X(t)), \quad t > 0,$$

is again an Itô process, and is given by

$$dY(t) = \frac{\partial g}{\partial t}(t, X(t))dt + \frac{\partial g}{\partial x}(t, X(t))dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(dX(t))^2, \tag{2.11}$$

where $(dX(t))^2 = (dX(t))(dX(t))$ is computed according to the rules

$$dtdt = dtdW(t) = dW(t)dt = 0, \quad dW(t)dW(t) = dt.$$

2.3 Stochastic Differential Equations

Consider the stochastic differential equation (SDE)

$$\begin{aligned} dX(t) &= f(t, X(t))dt + G(t, X(t))dW(t), \quad t > 0, \\ X(0) &= X_0, \end{aligned} \tag{2.12}$$

where $f(t,x)$ and $G(t,x)$ are scalar valued measurable deterministic functions for $t \in [0,T]$ $(0 < T < \infty)$ and $x \in R$ and prove the existence and uniqueness of the solution to (2.12) with respect to the given initial condition. $\{W(t), t \geq 0\}$ is a Wiener process, see Arnold, 1974, and Itô, 1951.

2.3.1 Existence and Uniqueness of a Solution

Theorem 2.7 [Gard, 1988, p. 68-69] If the following assumptions are satisfied:

(i) The functions $f(t,x)$ and $G(t,x)$ are defined for $t \in [0,T]$ and $x \in R$ and are measurable with respect to all their arguments:

(ii) There exists a constant $K > 0$ such that $t \in [0,T]$ and $x, y \in R$ such that

$$|f(t,x) - f(t,y)| - |G(t,x) - G(t,y)| \leq K|x-y|, \tag{2.13}$$

$$|f(t,x)|^2 + |G(t,x)|^2 \leq K^2(1+|x|^2); \tag{2.14}$$

(iii) X_0 does not depend on $W(t)$ and $E|X_0|^2 < \infty$.

Then there exists a solution of (2.12) defined on $[0,T]$ which is continuous w.p. 1 and such that

$$\sup_{0\leq t\leq T} E|X(t)|^2 < \infty.$$

Additionally, a solution with these properties is pathwise unique, in the sense that, if x and y are two solutions of (2.12)

$$P\Big\{\sup_{0\leq t\leq T} |X_1(t) - X_2(t)| = 0\Big\} = 1.$$

2.4 Poisson Jump processes

We consider the fundamental Poisson Jump process. This section picks details from Shreve, 1974, pp. 476-476.

Definition 2.18 [Applebaum, 2009, p. 43] Let $\{X(t), t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. We say that $X(t)$ is a Lévy process if:

(i) $X(0) = 0$, a.s.,

(ii) $X(t)$ has independent and stationary increments; stationary increments means that for t_1, t_2 in $[0,\infty)$, and $t_1 < t_2$, the distribution of $X(t_2) - X(t_1)$ is the same as that of $X(t_2+h) - X(t_1+h)$ for any $h > 0$,

(iii) $X(t)$ is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$,

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0.$$

Also, notice that (iii) is equivalent to the condition

$$\lim_{t \downarrow 0} P(|X(t)| > a) = 0.$$

for all $a > 0$ whenever (i) and (ii) hold.

Let us define a jump process as the sum of a nonrandom initial condition, a Riemman integral with respect to dt, an Itô integral with respect to a Brownian motion W(t), and a pure jump process. Hence, a jump process X (the integrator) will right be continuous and of the form

$$X(t) = X(0) + R(t) + I(t) + J(t). \tag{2.15}$$

In expression (2.15), X(0) is a nonrandom initial condition. The process $R(t) = \int_0^t \Theta(s)\,ds$ in (2.15) is a Riemman integral for some adapted process $\Theta(t)$. The process $I(t) = \int_0^t \Gamma(s)\,dW(s)$ is an Itô integral of an adapted process $\Gamma(t)$ with respect to a Brownian motion relative to the filtration $\mathcal{F}(t)$. The term $J(t)$ in (2.15) is an adapted, right-continuous pure jump process with $J(0) = 0$. It is right-continuous in the sense that $J(t) = \lim_{s \downarrow t} J(s)$ for all $t \geq 0$. The left-continuous version of $J(t)$ will be denoted $J(t-)$.

The continuous part of X is defined as

$$X^c(t) = X(0) + R(t) + I(t) = X(0) + \int_0^t \Theta(s)ds + \int_0^t \Gamma(s)dW(s).$$

The quadratic variation of this process is

$$[X^c, X^c](t) = \int_0^t \Gamma^2(s)ds,$$

which can be written in differential form as

$$dX^c(t)dX^c(t) = \Gamma^2(t)dt.$$

Definition 2.19 [Shreve, 2004, p. 475] A process $X(t)$ of the form (2.15) with Riemman integral part $R(t)$, with Itô integral $I(t)$, and a pure jump part $J(t)$ as previously described will be termed a jump process.

Let us define the stochastic integral given by

$$\int_0^t \Phi(s)dX(s),$$

where Φ and X to be defined below.

Definition 2.20 [Shreve, 2004, p. 475] (Lévy - Itô decomposition) Let $X(t)$ be a jump process of the form (2.15) and let $\Phi(s)$ be an adapted process. The stochastic integral of $\Phi(s)$ with respect to X is defined to be

$$\int_0^t \Phi(s)dX(s) = \int_0^t \Phi(s)\Theta(s)ds + \int_0^t \Phi(s)\Gamma(s)dW(s) + \sum_{0\leq s\leq t} \Phi(s)\Delta J(s). \tag{2.16}$$

In differential notation (2.16) can be written as

$$\begin{aligned} \Phi(t)dX(t) &= \Phi(t)dR(t) + \Phi(t)dI(t) + \Phi(t)dJ(t) \\ &= \Phi(t)dX^c(t) + \Phi(t)dJ(t), \end{aligned}$$

where

$$\begin{aligned}\Phi(t)dR(t) &= \Phi(t)\Theta(t)dt,\\ \Phi(t)dI(t) &= \Phi(t)\Gamma(t)dW(t),\\ \Phi(t)dX^c(t) &= \Phi(t)\Theta(t)dt+\Phi(t)\Gamma(t)dW(t).\end{aligned}$$

Theorem 2.8 [Shreve, 2004, p. 477] Assume that the jump process $X(s)$ described by (2.15) is a martingale, the integrand $\Phi(s)$ is left-continuous and adapted, and

$$E\int_0^t \Gamma^2(s)\Phi^2(s)ds<\infty \quad \text{for all t} \geq 0.$$

Then the stochastic integral $\int_0^t \Phi(s)dX(s)$ is also a martingale.

Theorem 2.9 [Shreve, 2004, p. 484] (Itô formula for a jump process) Let $X(t)$ be a jump process and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined and continuous, then

$$\begin{aligned}f(X(t)) &= f(X(0))+\int_0^t f'(X(s))dX^c(s)+\frac{1}{2}\int_0^t f''(X(s))dX^c(s)dX^c(s)\\ &+\sum_{0\leq s\leq t}[f(X(s))-f(X(s-))].\end{aligned}$$

Theorem 2.10 [Shreve, 2004, p. 484] (Two-dimensional Itô formula for processes with jumps) Let $X_1(t)$ and $X_2(t)$ be jump processes, and let $f(t,x_1,x_2)$ be a function whose first and second partial derivatives appearing in the following formula are defined and continuous. Then

$$\begin{aligned}f(t,X_1(t),X_2(t) &= f(0,X_1(0),X_2(0)+\int_0^t f_t(s,X_1(s),X_2(s))ds\\ &+\int_0^t f_{x_1}(s,X_1(s),X_2(s))dX_1^c(s)+\int_0^t f_{x_2}(s,X_1(s),X_2(s))dX_2^c(s)\\ &+\frac{1}{2}\int_0^t f_{x_1,x_1}(s,X_1(s),X_2(s))dX_1^c(s)dX_1^c(s)\\ &+\int_0^t f_{x_1,x_2}(s,X_1(s),X_2(s))dX_1^c(s)dX_2^c(s)\\ &+\frac{1}{2}\int_0^t f_{x_2,x_2}(s,X_1(s),X_2(s))dX_2^c(s)dX_2^c(s)\\ &+\sum_{0\leq s\leq t}[f(s,X_1(s),X_2(s))-f(s,X_1(s-)),X_2(s-))].\end{aligned} \tag{2.17}$$

Corollary 2.1. [Shreve, 2004, p. 489] (Itô's product rule for jump processes) Let $X_1(t)$ and $X_2(t)$ be jump processes. Then

$$X_1(t)X_2(t) = X_1(0)X_2(0) + \int_0^t X_2(s-)dX_1(s) + \int_0^t X_1(s-)dX_2(s) + [X_1, X_2](t).$$

Let $\mathbb{N}$ be the set of natural numbers.

Definition 2.21 [Applebaum, 2009, p. 49] The Poisson process $N(t)$ with intensity $\lambda > 0$ is a Lévy process taking values in $\mathbb{N} \cup \{0\}$ wherein each $N(t)$ has the probability function

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, 2, \ldots.$$

Note that the sample paths of $N(t)$ are piecewise constant on finite intervals with jump discontinuities of size 1 at each of the random times $(T_n, n \in \mathbb{N})$.

Let $U_1, U_2, \ldots$ be a sequence of independent and identicallly distributed (i.i.d.) random variables with mean $E[U_i] = \gamma$, $i = 1, 2, \ldots$. The random variables $U_1, U_2, \ldots$ are independent of the Poisson process $N(t)$.

Definition 2.22 [Shreve, 2004, p. 468] The Compound Poisson process is defined as

$$Q(t) = \sum_{i=1}^{N(t)} U_i, \quad t \geq 0.$$

Note that the jumps in $Q(t)$ are of random size while the jumps in N(t) are always of size 1. Also, $Q(t)$ jumps occur at the same time as $N(t)$ does.

For $0 \leq s < t$, the increment $Q(t) - Q(s)$ is independent of $\mathcal{F}(s)$ and has mean $\gamma\lambda(t-s)$. In fact

$$\begin{aligned} E[Q(t)|\mathcal{F}(s)] &= E[Q(t) - Q(s) + Q(s)|\mathcal{F}(s)] \\ &= E[Q(t) - Q(s)|\mathcal{F}(s)] + Q(s) \\ &= \gamma\lambda(t-s) + Q(s) \end{aligned}$$

Hence the compound Poisson process is not a martingale.

Theorem 2.11 [Applebaum, 2009, p. 49] The compound Poisson process $\{Q(t), t \geq 0\}$ is a Lévy process.

Proof Conditions (i) and (ii) follow immediately. To obtain (iii), let $a > 0$, next by conditioning and independence

$$P(|Q(t)| > a) = \sum_{n=0}^{\infty} P(|Z(1) + \cdots + Z(n)| > a) P(N(t) = n),$$

by dominated convergence theorem [Arnold, 1974, p. 11] (iii) follows. □

Theorem 2.12 [Shreve, 2004, p. 470] Let $\{Q(t), t \geq 0\}$ be the compound Poisson process. Then the compensated compound Poisson process

$$Q(t) - \gamma\lambda t$$

is a martingale.

Proof. Let $0 \leq s < t$ be given. Taking $E[Q(t)] = \gamma\lambda t$ and following the previous argument we observe that $E[Q(t) - \gamma\lambda t|\mathcal{F}(s)] = Q(s) - \gamma\lambda s$. □

Lemma 2.2 [Svishchuk and Kalemanova, 2000] Let $\{Y_n, n \geq 1\}$ be a sequence of i.i.d nonnegative random variables, $E[Y_n] < +\infty$, and $\{N(t), t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$, independent of $\{Y_n, n \geq 1\}$. Then

$$E\Big[\prod_{n=1}^{N(t)} Y_n\Big] = \exp\{\lambda t(E[Y_1] - 1)\}, \quad \text{for all } t \geq 0.$$

2.5 Probabilistic Properties: Stochastic Stability

In this Section, we collect stochastic stability properties from Svishchuk and Kalemanova, 2000, and Govindan and Acosta, 2008, and Khasminskii, 2012.

Let $f(t,0) = 0$ and $g(t,0) = 0$ a.e. t, so that equation (2.12) admits the trivial solution $X(t) \equiv 0$. Let $X(t) = X(t, X_0)$ a solution of (2.12), where $X_0 \in R$ is a constant (sometimes, we shall consider X_0 as random variable independent of $\{W(t), t \geq 0\}$, with $E|X_0|^2 < \infty$. Let $X^*(t, X_0^*)$ be any other solution of (2.12) with $X^*(0) = X_0^*$.

Definition 2.23 The solution $X(t)$ of (2.12) is called exponentially mean square stable if there exists positive constants C and γ such that

$$E|X(t) - X^*(t)|^2 \leq CE|X_0 - X_0^*|^2 e^{-\gamma t}, \quad t \geq 0.$$

Definition 2.24 The solution $X(t)$ of (2.12) is bounded in probability uniformly in t if

$$\sup_{t\geq 0} P\{|X(t)| > M\} \to 0, \quad \text{as} \quad \mathrm{M} \to \infty,$$

for sufficiently small $|X_0|$.

The following lemma will be needed.

Lemma 2.3 (Gronwall-Bellman inequality) [Khasminskii, 2012, p. 9] Let $u(t)$ and $v(t)$ be non-negative functions and let k be a positive constant such that for $t \geq s$,

$$u(t) \leq k + \int_s^t u(\tau)v(\tau)d\tau.$$

Then for $t \geq s$,

$$u(t) \leq k \exp\left\{\int_s^t v(\tau)d\tau\right\}.$$

Chapter 3

An Alternative to Itô's Formula to Solve some Interest Rate Models

In this Section, we solve several financial models using classical Itô's formula and also develop a method using basic ordinary differential equations (ODEs).

The proofs of all the theorems in this chapter appear to be new.

3.1 A Linear Ordinary Differential Equation

Definition 3.1 A first-order stochastic differential equation of the form

$$a_1(t)\frac{dy}{dt} + a_0(t)y = h(t,\omega), \quad t > 0, \tag{3.1}$$

is said to be a linear equation in the dependent variable y, where $a_0(t)$ and $a_1(t)$ are real-valued continuous functions of t and $h(t,\omega)$ is a continuous stochastic process almost surely. We suppose that $a_1(t) \neq 0$ for all t. By dividing both sides of (3.1) by the lead coefficient $a_1(t)$, we obtain the standard form of a first-order stochastic differential equation

$$\frac{dy}{dt} + P(t)y = f(t,\omega), \tag{3.2}$$

where $P(t)$ is a real-valued continuous functions of t and $f(t,\omega)$ is a continuous stochastic process almost surely. We seek a solution of (3.2) on $t > 0$.

Method of solution of the equation (3.2) Following the linear ODE method, see Zill and Cullen, 2004, p. 55, and Ladde and Ladde, 2012, p. 122, we introduce the following steps:

(i) Put a linear equation of the form (3.1) into the standard form (3.2).

(ii) From the standard form, identify $P(t)$ and then find the integrating factor $e^{\int P(t)dt}$.

(iii) Multiply the standard form (3.2) by this integrating factor. The LHS of the resulting equation is nothing but the derivative of the product $e^{\int P(t)dt}y$, as follows:

$$\frac{d}{dt}[e^{\int P(t)dt}y] = e^{\int P(t)dt}f(t,\omega). \tag{3.3}$$

(iv) Integrating both sides of the equation (3.3) with respect to t yields the solution, provided the integrals on the RHS are well-defined.

Remark 3.1 Notice that we take $f(t,\omega) = h_1(t) + h_2(t)\xi(t)$, where $h_1(t)$ and $h_2(t)$ are continuous real-valued functions in the application of this method in the following sections. We call this method as a linear ODE method.

3.2 Vasicek

In this section, we begin our study with standard interest rate models.

The Vasicek model, see Vasicek,1977, for the interest rate process $r(t)$ is given by the SDE:

$$\begin{aligned} dr(t) &= [\alpha - \beta r(t)]dt + \sigma dW(t), \quad t > 0, \\ r(0) &= r_0, \end{aligned} \tag{3.4}$$

where α, β and σ are positive constants.

Theorem 3.1 Equation (3.4) has a solution given by

$$r(t) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW(s).$$

First Proof Our approach to find the solution of equation (3.4) is by applying the linear ODE method. Rewriting (3.4) as

$$\frac{dr(t)}{dt} + \beta r(t) = \alpha + \sigma \frac{dW(t)}{dt}, \quad t > 0.$$

That is

$$\frac{dr(t)}{dt} + \beta r(t) = \alpha + \sigma \xi(t), \quad t > 0. \tag{3.5}$$

Equation (3.5) is in the form of the first-order ordinary differential equation (3.2) with $P(t) = \beta$. Its integrating factor is $e^{\int \beta dt} = e^{\beta t}$. Multiplying both sides of equation (3.5) by this integrating factor, it follows that

$$e^{\beta t} \frac{dr(t)}{dt} + e^{\beta t} \beta t r(t) = e^{\beta t} \alpha + e^{\beta t} \sigma \xi(t).$$

Thus

$$\frac{d}{dt}\Big(e^{\beta t} r(t)\Big) = e^{\beta t} \alpha + e^{\beta t} \sigma \xi(t). \tag{3.6}$$

Integrating (3.6) from 0 to t, we have

$$e^{\beta t} r(t) \quad = \quad r_0 + \alpha \int_0^t e^{\beta s} ds + \sigma \int_0^t e^{\beta s} dW(s).$$

Hence, the solution of equation (3.4) is given by

$$r(t) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dW(s). \quad \square \tag{3.7}$$

Second Proof Interestingly, an application of Itô's formula (Theorem 2.6) to the SDE (3.4) would also produce the same solution (3.7) as shown below:

To do so, we pick the function $g(t,r) = e^{\beta t} r$, where $r(t)$ is the solution process of equation (3.4). Applying Itô's formula, we get

$$\begin{aligned} d(e^{\beta t} r(t)) &= \frac{\partial g}{\partial t}(t, r(t))dt + \frac{\partial g}{\partial r}(t, r(t))dr(t) + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}(t, r(t))dr(t)dr(t) \\ &= \beta e^{\beta t} r(t)dt + e^{\beta t} dr(t) + \frac{1}{2}(0)dr(t)dr(t) \\ &= \beta e^{\beta t} r(t)dt + e^{\beta t}\Big([\alpha - \beta r(t)]dt + \sigma dW(t)\Big) \\ &= \beta e^{\beta t} r(t)dt + e^{\beta t}\alpha dt - \beta e^{\beta t} r(t)dt + e^{\beta t}\sigma dW(t) \\ &= e^{\beta t}\alpha dt + \sigma e^{\beta t} dW(t). \end{aligned} \tag{3.8}$$

Integrating equation (3.8) from 0 to t, we have

$$r(t) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s} dW(s),$$

which coincides with (3.7). $\square$

3.3 Cox-Ingersoll-Ross

The Cox-Ingersoll-Ross (CIR) model, see Cox, Ingersoll, and Ross, 1985b, for the interest rate process $r(t)$ is given by the SDE:

$$\begin{aligned} dr(t) &= [\alpha - \beta r(t)]dt + \sigma\sqrt{r(t)}dW(t), \quad t > 0, \\ r(0) &= r_0, \end{aligned} \tag{3.9}$$

where α, β and σ are positive constants.

Theorem 3.2 Equation (3.9) has a solution that satisfies the following integral equation

$$r(t) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\sqrt{r(s)}dW(s).$$

Proof Proceeding as in Section 3.2, we have

$$\frac{dr(t)}{dt} + \beta r(t) = \alpha + \sigma\sqrt{r(t)}\frac{dW(t)}{dt}, \quad t > 0.$$

That is

$$\frac{dr(t)}{dt} + \beta r(t) = \alpha + \sigma\sqrt{r(t)}\xi(t), \quad t > 0. \tag{3.10}$$

Equation (3.10) is in the form of the first-order ordinary differential equation (3.2) and its integrating factor is $e^{\int \beta dt} = e^{\beta t}$. Multiplying both sides of equation (3.10) by this integrating factor, it follows that

$$e^{\beta t}\frac{dr(t)}{dt} + e^{\beta t}\beta r(t) = e^{\beta t}\alpha + e^{\beta t}\sigma\sqrt{r(t)}\xi(t).$$

Thus

$$\frac{d}{dt}\Big(e^{\beta t}r(t)\Big) = e^{\beta t}\alpha + e^{\beta t}\sigma\sqrt{r(t)}\xi(t). \tag{3.11}$$

Integrating (3.11) from 0 to t, we have

$$r(t) = e^{-\beta t}r_0 + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\sqrt{r(s)}dW(s). \quad \Box \tag{3.12}$$

Remark 3.2 Interestingly, an application of Itô's formula (Theorem 2.6) to the SDE (3.9) would also produce the same solution (3.12).

3.4 Hull-White

The Hull-White interest rate model also termed as extended Vasicek model for the interest rate process $r(t)$ is given by the SDE:

$$\begin{aligned} dr(t) &= [\alpha(t) - \beta(t)r(t)]dt + \sigma(t)dW(t), \quad t > 0, \qquad (3.13)\\ r(0) &= r_0, \end{aligned}$$

where $\alpha(t)$, $\beta(t)$ and $\sigma(t)$ are nonrandom positive functions of the time variable t.

Theorem 3.3 Equation (3.13) has a solution given by

$$r(t) = e^{-k(t)}r_0 + \int_0^t e^{-[k(t)-k(s)]}\alpha(s)ds + \int_0^t e^{-[k(t)-k(s)]}\sigma(s)dW(s),$$

where $k(t) = \int \beta(t)dt$.

Proof Proceeding as before, we obtain

$$\frac{dr(t)}{dt} + \beta(t)r(t) = \alpha(t) + \sigma(t)\xi(t), \quad t > 0. \tag{3.14}$$

Equation (3.14) is in the form of the first-order ordinary differential equation (3.2) and its integrating factor is $e^{\int \beta(t)dt} = e^{k(t)}$. Multiplying both sides of equation (3.14) by this integrating factor, it follows that

$$e^{k(t)}\frac{dr(t)}{dt} + e^{k(t)}\beta(t)r(t) = e^{k(t)}\alpha(t) + e^{k(t)}\sigma(t)\xi(t).$$

Thus

$$\frac{d}{dt}\Big(e^{k(t)}r(t)\Big) = e^{k(t)}\alpha(t) + e^{k(t)}\sigma(t)\xi(t). \tag{3.15}$$

Integrating (3.15) from 0 to t, we have the solution

$$r(t) = e^{-k(t)}r_0 + \int_0^t e^{-[k(t)-k(s)]}\alpha(s)ds + \int_0^t e^{-[k(t)-k(s)]}\sigma(s)dW(s). \quad \square \tag{3.16}$$

Remark 3.3 An application of Itô's formula would also produce the same solution.

3.4.1 Extended Hull-White

Consider an extension of the Hull-White model for the interest rate process $r(t)$ given by the SDE:

$$\begin{aligned} dr(t) &= k[\theta(t) - r(t)]dt + \sigma r(t)dW(t), \quad t > 0, \\ r(0) &= r_0, \end{aligned} \tag{3.17}$$

where k and σ are positive constants and $\theta(t)$ is a nonrandom positive function of the time variable t.

Theorem 3.4 Equation (3.17) has a solution that satisfies the following integral equation

$$r(t) = r_0 e^{-kt} + \int_0^t e^{-k(t-s)} k\theta(s)ds + \int_0^t e^{-k(t-s)} \sigma r(s)dW(s).$$

Proof Rewriting (3.17) as a first-order ordinary differential equation:

$$\frac{dr(t)}{dt} + kr(t) = k\theta(t) + \sigma r(t)\xi(t), \quad t > 0 \tag{3.18}$$

which is in the form of the first-order ordinary differential equation (3.2). Its integrating factor is e^{kt}. Multiplying both sides of equation (3.18) by this integrating factor, it follows that

$$e^{kt}\frac{dr(t)}{dt} + e^{kt}kr(t) = e^{kt}k\theta(t) + e^{kt}\sigma r(t)\xi(t).$$

Thus

$$\frac{d}{dt}\left(e^{kt}r(t)\right) = e^{kt}k\theta(t) + e^{kt}\sigma r(t)\xi(t). \tag{3.19}$$

Integrating (3.19) from 0 to t, the solution of equation (3.17) is given by

$$r(t) = r_0 e^{-kt} + \int_0^t e^{-k(t-s)} k\theta(s)ds + \int_0^t e^{-k(t-s)} \sigma r(s)dW(s). \quad \square \tag{3.20}$$

Remark 3.4 An application of Itô's formula would also produce the same solution.

3.4.2 Hull-White: A CIR Style Extension

A CIR style extension to a Hull-White class model for the interest rate process $r(t)$ is given by the SDE:

$$\begin{aligned} dr(t) &= k[\theta(t) - r(t)]dt + \sigma\sqrt{r(t)}dW(t), \quad t > 0, \\ r(0) &= r_0, \end{aligned} \tag{3.21}$$

where k and σ are positive constants and $\theta(t)$ is a nonrandom positive function of the time variable t.

Theorem 3.5 Equation (3.21) has a solution that satisfies the following integral equation

$$r(t) = e^{-kt}r_0 + \int_0^t e^{-k(t-s)}k\theta(s)ds + \int_0^t e^{-k(t-s)}\sigma\sqrt{r(s)}dW(s).$$

Proof From (3.21), we obtain

$$\frac{dr(t)}{dt} + kr(t) = k\theta(t) + \sigma\sqrt{r(t)}\xi(t), \quad t > 0.$$

Thus

$$\frac{d}{dt}\left(e^{kt}r(t)\right) = e^{kt}k\theta(t) + e^{kt}\sigma\sqrt{r(t)}\xi(t). \tag{3.22}$$

Integrating (3.22) from 0 to t, the solution of equation (3.21) is given by

$$r(t) = e^{-kt}r_0 + \int_0^t e^{-k(t-s)}k\theta(s)ds + \int_0^t e^{-k(t-s)}\sigma\sqrt{r(s)}dW(s). \quad \Box \tag{3.23}$$

Remark 3.5 An application of Itô's formula would also produce the same solution.

3.5 Brennan-Schwartz

An extension of the Vasicek interest rate process $r(t)$ with a multiplicative difussion, known as the Brennan-Schwartz interest rate model, is given by the SDE:

$$
\begin{aligned}
dr(t) &= k[\theta - r(t)]dt + \sigma r(t) dW(t), \quad t > 0, \\
r(0) &= r_0,
\end{aligned}
\tag{3.24}
$$

where k, θ and σ are positive constants.

Theorem 3.6 Equation (3.24) has a solution that satisfies the following integral equation

$$r(t) = r_0 e^{-kt} + \theta(1 - e^{-kt}) + \int_0^t e^{-k(t-s)} \sigma r(s) dW(s).$$

Proof As before,

$$\frac{dr(t)}{dt} + kr(t) = k\theta + \sigma r(t)\xi(t), \quad t > 0.$$

Thus

$$\frac{d}{dt}\left(e^{kt} r(t)\right) = e^{kt} k\theta + e^{kt} \sigma r(t) \xi(t). \tag{3.25}$$

Integrating (3.25) from 0 to t, the solution of equation (3.24) satisfies

$$r(t) = r_0 e^{-kt} + \theta(1 - e^{-kt}) + \int_0^t e^{-k(t-s)} \sigma r(s) dW(s). \quad \square \tag{3.26}$$

Remark 3.6 An application of Itô's formula would also produce the same solution.

3.6 Exponential Vasicek

A natural way to obtain a lognormal short-rate model is by assuming that the logarithm of r follows an Ornstein-Uhlenbeck process y, so that

$$r(t) = e^{y(t)}, \quad t > 0, \tag{3.27}$$

$$dy(t) = [\theta - ay(t)]dt + \sigma dW(t), \quad t > 0, \tag{3.28}$$

$$y(0) = y_0,$$

where θ, a and σ are positive constants and y_0 is a real number.

Since the short-rate is defined as the exponential of a process that is equivalent to that of Vasicek, this model is referred as the exponential-Vasicek interest rate model.

Theorem 3.7 Equation (3.27) has a solution given by

$$r(t) = \exp\Big\{\ln(r_0)e^{-at} + \frac{\theta}{a}\Big[1 - e^{-at}\Big] + \sigma e^{-at}\int_0^t e^{au}dW(u)\Big\}.$$

Proof After rewriting (3.28) as a first-order ordinary differential equation, and solving for $y(t)$ yields

$$y(t) = y_0 e^{-at} + \frac{\theta}{a}(1 - e^{-at}) + \sigma e^{-at}\int_0^t e^{as}dW(s).$$

Hence, the solution process to the exponential Vasicek interest rate model from (3.27) is

$$r(t) = \exp\Big\{\ln(r_0)e^{-at} + \frac{\theta}{a}\Big[1 - e^{-at}\Big] + \sigma e^{-at}\int_0^t e^{au}dW(u)\Big\}. \quad \square \tag{3.29}$$

Remark 3.7 An application of Itô's formula, with the function $g(t, y) = e^{at}y$, would also produce the same solution.

3.7 Dothan

We consider the following form of the Dothan model for the interest rate process $r(t)$:

$$\begin{aligned} dr(t) &= ar(t)dt + \sigma r(t)dW(t), \quad t > 0, \\ r(0) &= r_0, \end{aligned} \tag{3.30}$$

where a is a real constant and σ is positive constant.

Theorem 3.8 Equation (3.30) has a solution that satisfies the following integral equation

$$r(t) = r_0 e^{at} + e^{at}\sigma \int_0^t e^{-as} r(s)dW(s).$$

Proof Clearly, the linear ODE method yields the following form for the solution process

$$r(t) = r_0 e^{at} + e^{at}\sigma \int_0^t e^{-as} r(s)dW(s). \quad \square \tag{3.31}$$

Remark 3.8 An application of Itô's formula, with the function $g(t,r) = \ln r$, to equation (3.30) gives the following well-known explicit solution process

$$r(t) = r_0 \exp\Big\{(a - \frac{1}{2}\sigma^2)t + \sigma W(t)\Big\}, \quad t \geq 0. \tag{3.32}$$

3.8 A Generalized Model

We consider a generalized interest rate model, see Svishchuk and Kalemanova, 2000, from which the Cox-Ingersoll-Ross interest rate model, and the Brennan-Schwartz interest rate model, can be seen as special cases for an appropriate choice of the parameter δ:

$$\begin{aligned} r(t) &= [\alpha - \beta r(t)]dt + \sigma r^{\delta}(t)dW(t), \quad t > 0, \\ r(0) &= r_0, \end{aligned} \tag{3.33}$$

where r_0, α, β and σ are positive constants and $\delta \in [\frac{1}{2}, \infty)$.

Theorem 3.9 Equation (3.33) has a solution that satisfies the following integral equation

$$r(t) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + e^{-\beta t}\sigma \int_0^t e^{\beta s} r^{\delta}(t) dW(s).$$

Proof It can be easily seen that $r(t)$ satisfies

$$r(t) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + e^{-\beta t}\sigma \int_0^t e^{\beta s} r^{\delta}(t) dW(s). \tag{3.34}$$

is the solution of the SDE (3.33) by application of the linear ODE method. □

Remark 3.9 An application of Itô's formula would also produce the same solution.

3.9 Some More Models

The Mercurio-Moraleda interest rate model, see Brigo and Mercurio, 2007, p. 57 is described as follows:

$$\begin{aligned} dr(t) &= r(t)\left[\eta(t) - \left(\lambda - \frac{\gamma}{1+\gamma t}\right)\ln r(t)\right]dt + \sigma r(t) dW(t), \quad t > 0, \\ r(0) &= r_0, \end{aligned} \tag{3.35}$$

where λ, γ and σ are positive constants.

Theorem 3.10 The Mercurio-Moraleda interest rate model (3.35) has a solution given by

$$r(t) = \exp\left\{e^{-k(t)} \ln r_0 + \int_0^t e^{-[k(t)-k(s)]}\eta(s) ds + \int_0^t e^{-[k(t)-k(s)]}\sigma dW(s)\right\},$$

where $k(t) = \int \beta(t) dt$, and $\beta(t) = \left(\lambda - \frac{\gamma}{1+\gamma t}\right)$.

Proof Note that (3.35) is the Hull-White model (3.13) expressed in a logarithmic form with $\beta(t) = \left(\lambda - \frac{\gamma}{1+\gamma t}\right)$. The proof follows as in Theorem 3.3.

The Extended Exponential Vasicek interest model, see Brigo and Mercurio, 2007, p. 57 as follows:

$$\begin{aligned} r(t) &= x(t)+\varphi(t), \quad t>0, && (3.36)\\ dx(t) &= x(t)[\eta(t)-a\ln x(t)]dt+\sigma x(t)dW(t), \quad t>0, && (3.37)\\ r(0) &= r_0,\\ x(0) &= x_0, \end{aligned}$$

where $\varphi(t)$ is a shift deterministic function of time, and a and σ are positive constants.

Theorem 3.11 The solution process to the Extended Exponential Vasicek interest rate model (3.36)-(3.37) is given explicitly by

$$\begin{aligned} r(t) &= x(t)+\varphi(t), \quad t>0, && (3.38)\\ x(t) &= \exp\left\{\ln(x_0)e^{-at}+\frac{\theta}{a}\left[1-e^{-at}\right]+\sigma e^{-at}\int_0^t e^{au}dW(u)\right\}, \quad t>0. && (3.39) \end{aligned}$$

Proof It follows immediately from the proof of Theorem 3.7.

3.9.1 Brownian Bridge

The Brownian bridge process $Y(t)$ is given by the SDE:

$$\begin{aligned} dY(t) &= \frac{b-Y(t)}{1-t}dt+dW(t), \quad 0\le t<1, && (3.40)\\ Y(0) &= Y_0, \end{aligned}$$

where $b\in R$.

Theorem 3.12 Equation (3.40) has a solution given by

$$Y(t) = (1-t)Y_0+bt+(1-t)\int_0^t (1-s)^{-1}dW(s).$$

Proof Our approach to finding the solution of equation (3.40) requires rewriting it as

$$\frac{dY(t)}{dt} + \frac{1}{1-t}Y(t) = \frac{b}{1-t} + \xi(t), \quad 0 \le t < 1.$$

The integrating factor is $e^{\int \frac{1}{1-t}dt} = (1-t)^{-1}$. Thus

$$\frac{d}{dt}\Big((1-t)^{-1}Y(t)\Big) = (1-t)^{-1}\frac{b}{1-t} + (1-t)^{-1}\xi(t). \tag{3.41}$$

Integrating (3.41) from 0 to t,

$$(1-t)^{-1}Y(t) \quad = \quad Y_0 + \int_0^t (1-s)^{-1}\frac{b}{1-s}ds + \int_0^t (1-s)^{-1}dW(s).$$

Hence, the solution of equation (3.40) is given by

$$Y(t) \quad = \quad (1-t)Y_0 + bt + (1-t)\int_0^t (1-s)^{-1}dW(s). \quad \Box \tag{3.42}$$

Remark 3.10 An application of Itô's formula would also produce the same solution.

To give a counter example, consider the following generalized Vasicek model

$$\begin{aligned} dr(t) &= [\alpha - \beta r(t)]^{\gamma}dt + \sigma dW(t), \quad t > 0, \qquad (3.43) \\ r(0) &= r_0, \end{aligned}$$

where α, β and σ are positive constants and $0 < \gamma \le 1$. When $\gamma = 1$ it is the Vasicek model. Note that when $\gamma = 1/2$ the linear ODE method does not apply.

Chapter 4

Probabilistic Properties of some Interest Rate Models with Poisson Jumps

Stability of stochastic differential equations is a well-established area of research, see, for instance, Khasminskii, 2012. However, the stability of financial models has not received considerable attention in the literature. This perhaps motivated Svishchuk and Kalemanova, 2000, to initiate a study on stability properties like stability in probability and p-stability of solutions of several interesting financial models like Black-Scholes, Vasicek, and Cox-Ingersoll-Ross and also such models with Poisson jumps. Subsequently, Govindan and Acosta, 2008, continued the study further by considering uniformly boundedness in probability and exponential mean-square stability of financial models mentioned earlier. Moreover, Bhan and Mandrekar, 2010, considered some recurrence properties of term structure models. We refer to Brigo and Mercurio, 2007, for a study on interest rate models.

Motivated by these works, we obtain exponential mean square stability, boundedness in probability uniformly in t, and asymptotic quadratic mean of some standard interest rate models, including such models with Poisson jumps. We wll also obtain moment properties of some of these models.

4.1 Interest Rate Models

We begin with the Vasicek model.

4.1.1 Vasicek

Theorem 4.1 The solution of the Vasicek interest rate model (3.4) satisfies

$$E|r(t)|^p \le 3^{p-1}\Big\{E|r_0|^p + \frac{\alpha^p}{\beta^p} + \Big[\frac{1}{2}p(p-1)\Big]^{p/2} T^{p/2-1}\frac{\sigma^p}{\beta p}(e^{\beta p T}-1)\Big\}, \tag{4.1}$$

for $p \ge 2$ and for each $t \in [0,T]$.

Proof Consider

$$r(t) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1-e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\,dW(s).$$

Using the following well-known inequality

$$|a+b+c|^p \le 3^{p-1}\{|a|^p + |b|^p + |c|^p\},$$

the following property for the stochastic integral

$$E\Big|\int_0^t G(s)\,dW(s)\Big|^p \le \Big[\frac{1}{2}p(p-1)\Big]^{p/2} T^{p/2-1}\int_0^t E|G(s)|^p\,ds, \quad t\in[0,T], \quad p\ge 2,$$

and taking expectation, we have

$$\begin{aligned}
E|r(t)|^p &\le 3^{p-1}\Big\{e^{-\beta t p}E|r_0|^p + E\Big|\frac{\alpha}{\beta}(1-e^{-\beta t})\Big|^p + \sigma^p e^{-\beta t p}E\Big|\int_0^t e^{\beta s}\,dW(s)\Big|^p\Big\}\\
&\le 3^{p-1}\Big\{E|r_0|^p + \frac{\alpha^p}{\beta^p}(1-e^{-\beta t})^p + \\
&\qquad \Big[\frac{1}{2}p(p-1)\Big]^{p/2} T^{p/2-1}\sigma^p\int_0^t e^{\beta s p}\,ds\Big\},
\end{aligned}$$

from which (4.1) follows. □

4.1.2 Brennan-Schwartz

This is an extension of the Vasicek interest rate process $r(t)$ with the volatility term assumed to be dependent on the interest rate level.

The following properties appear to be new.

Proposition 4.1 For the Brennan-Schwartz interest rate model (3.24), we have

$$\begin{aligned} E[r(t)] &= \theta, \qquad \text{as} \quad \mathrm{t} \to \infty, \quad \text{and} \\ Var[r(t)] &= \sigma^2\theta^2/2k, \quad \text{as} \quad \mathrm{t} \to \infty. \end{aligned}$$

Proof Taking expectation of the process (3.26) it follows:

$$\begin{aligned} E[r(t)] &= \theta + (r_0 - \theta)e^{-kt} \\ \lim_{t\to\infty} E[r(t)] &= \theta. \end{aligned}$$

For the second moment, we have

$$\begin{aligned} E[r^2(t)] &= \theta^2 + (r_0-\theta)^2 e^{-2kt} + 2\theta(r_0-\theta)e^{-kt} \\ &\quad + E\Big[e^{-2kt}\sigma^2 \int_0^t e^{2ku}\Big(\theta + (r_0-\theta)e^{-ku} \\ &\quad + e^{-ku}\sigma \int_0^u e^{kv} r(v)\,dW(v)\Big)^2 du\Big]. \end{aligned}$$

Taking limit

$$\begin{aligned} \lim_{t\to\infty} E[r^2(t)] &= \theta^2 + \\ &\quad \lim_{t\to\infty} E\,\frac{1}{e^{2kt}}\Big\{\sigma^2 \int_0^t e^{2ku}[\theta + (r_0-\theta)e^{-ku} + \\ &\quad e^{-ku}\sigma \int_0^u e^{kv} r(v)\,dW(v)]^2 du\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \theta^2 + \\
&\quad \lim_{t\to\infty} E\Big\{\frac{\sigma^2 \int_0^t e^{2ku}[\theta + (r_0-\theta)e^{-ku}]^2 du}{e^{2kt}}\Big\} \\
&= \theta^2 + \\
&\quad \lim_{t\to\infty} E\Big\{\frac{\sigma^2 e^{2kt}[\theta + (r_0-\theta)e^{-kt}]^2}{2ke^{2kt}}\Big\} \\
&= \theta^2 + \frac{\sigma^2\theta^2}{2k}. \quad \square
\end{aligned}
$$

Proposition 4.2 The Brennan-Schwartz interest rate model (3.24) is exponentially mean square stable provided $k > \sigma^2$.

Proof Let $r^*(t)$ be a solution of (3.24) satisfying $r^*(0) = r_0^*$. Then

$$r(t) - r^*(t) = (r_0 - r_0^*)e^{-kt} + \int_0^t e^{-k(t-s)}\sigma[r(s) - r^*(s)]dW(s).$$

Thus

$$e^{2kt}|r(t) - r^*(t)|^2 \leq 2|(r_0 - r_0^*)|^2 + 2\Big|\int_0^t e^{2ks}\sigma[r(s) - r^*(s)]dW(s)\Big|^2.$$

Taking expectation on both sides of this last expression, we get

$$e^{2kt}E|r(t) - r^*(t)|^2 \leq 2E|(r_0 - r_0^*)|^2 + \int_0^t 2\sigma^2 e^{2ks}E|r(s) - r^*(s)|^2 ds. \tag{4.2}$$

Now, by application of Gronwall's inequality, Lemma 2.3, to equation (4.2) we obtain

$$E|r(t) - r^*(t)|^2 \leq 2E|r_0 - r_0^*|^2 \exp\{-2(k - \sigma^2)t\}. \quad \square$$

Theorem 4.2 The solution of the Brennan-Schwartz interest rate model (3.24) satisfies

$$E|r(t)|^p \leq 3^{p-1}\Big\{E|r_0|^p + \theta^p e^{kpT}\Big\}e^{(L-kp)T}, \quad p \geq 2, \tag{4.3}$$

for each $t \in [0, T]$ where $L = 3^{p-1}\sigma^p\left[\frac{1}{2}p(p-1)\right]^{p/2}T^{p/2-1}$.

Proof From (3.26), it follows that

$$\begin{aligned} E|r(t)|^p &\leq 3^{p-1}\Big\{e^{-ktp}E|r_0|^p + \theta^p(1-e^{-kt})^p \\ &\quad + \Big[\frac{1}{2}p(p-1)\Big]^{p/2} T^{p/2-1}\int_0^t e^{-kp(t-s)}\sigma^p E|r(s)|^p ds\Big\}. \end{aligned}$$

Thus

$$\begin{aligned} e^{ktp}E|r(t)|^p &\leq 3^{p-1}\Big\{E|r_0|^p + \theta^p e^{ktp}(1-e^{-kt})^p\Big\} \\ &\quad + L\int_0^t e^{ksp}E|r(s)|^p ds. \end{aligned}$$

Applying Gronwall's inequality, Lemma 2.3, the result follows. □

Theorem 4.3 For the Brennan-Schwartz interest rate model (3.24), the solution processs is bounded in probability uniformly in t provided $2k > \sigma^2$.

Proof Consider the generating operator L, see Khasminskii, 2012, applied to the function $V(r) = r^2$, $r > 0$:

$$\begin{aligned} L(V(r)) &= V_r f(r) + \frac{1}{2}g^2(r)V_{rr}(r) \\ &= 2rk(\theta - r) + \frac{1}{2}\sigma^2 r^2 2 = 2k\theta r - 2kr^2 + \sigma^2 r^2. \end{aligned}$$

We have

$$L(V(r)) \leq 0 \quad \text{for} \quad \mathrm{r} \geq 2\mathrm{k}\theta/(2\mathrm{k}-\sigma^2) = \mathrm{C} > 0. \tag{4.4}$$

Now, from Itô's formula for $V(r) = r^2$, $r > 0$ and from (4.4), we get

$$E|r(t)|^2 = E[V(r(t))] = E[V(r_0)] + \int_0^t EL(Vr(u))du \leq E[V(r_0)] \leq E|r_0|^2.$$

From the last inequality, if $M \geq C$, then

$$P\{|r(t)| \geq M\} \leq \frac{1}{M^2} E|r(t)|^2 \leq \frac{1}{M^2} E|r_0|^2 \to 0, \quad \text{when} \quad \mathrm{M} \to \infty.$$

for sufficiently small r_0. $\square$

4.1.3 Hull-White

The Hull-White interest rate model, see Brigo and Mercurio, 2007, p. 72, is an arbitrage-free model that can be seen as the Vasicek interest rate model with a time-dependent reversion level. We first look at a general case and next we look at the Hull-White interest rate model when the time-dependent reversal level is described by considering the forward rate and a convenient expression for the volatility. In this case, it provides an exact calibration to the current term structure of interest rates.

Proposition 4.3 For the solution process (3.16),

$$\begin{aligned} E[r(t)] &= e^{-k(t)} r_0 + e^{-k(t)} \int_0^t e^{k(s)} \alpha(s) ds, \quad \text{and} \\ Var[r(t)] &= e^{-2k(t)} \int_0^t e^{2k(s)} \sigma^2(s) ds. \end{aligned}$$

Proof We take expectation and variance to the solution process (3.16) as follows:

$$\begin{aligned} E[r(t)] &= E\Big[e^{-k(t)} r_0 + e^{-k(t)} \int_0^t e^{k(s)} \alpha(s) ds + e^{-k(t)} \int_0^t e^{k(s)} \sigma(s) dW(s)\Big] \\ &= e^{-k(t)} r_0 + e^{-k(t)} \int_0^t e^{k(s)} \alpha(s) ds, \text{ and} \\ Var[r(t)] &= Var\Big[e^{-k(t)} r_0 + e^{-k(t)} \int_0^t e^{k(s)} \alpha(s) ds + e^{-k(t)} \int_0^t e^{k(s)} \sigma(s) dW(s)\Big] \\ &= Var\Big[e^{-k(t)} \int_0^t e^{k(s)} \sigma(s) dW(s)\Big] \\ &= e^{-2k(t)} \int_0^t e^{2k(s)} \sigma^2(s) ds. \quad \square \end{aligned}$$

A special case An exact fit to the current term structure of interest rates can be obtained with the following SDE:

$$\begin{aligned} dr(t) &= [\theta(t) - ar(t)]dt + \sigma dW(t), \quad t > 0, \\ r(0) &= r_0, \end{aligned} \tag{4.5}$$

where r_0, a and σ are positive constants. The nonrandom positive function $\theta(t)$ of the time variable t can be expressed as, see Hull, 2012, p. 692,

$$\theta(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \tag{4.6}$$

where $F(0,t)$ is the forward rate and $F_t(0,t)$ is the time derivative of the forward rate. The volatility is assumed to be of the form $v(u,t) = \frac{\sigma}{a}(1 - e^{-a(t-u)})$.

Proposition 4.4 The process

$$r(t) = r_0 e^{-at} + \alpha(t) + \sigma e^{-at}\int_0^t e^{au} dW(u), \tag{4.7}$$

where

$$\alpha(t) = F(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$$

is the solution of the SDE (4.5).

Proof It follows by applying Itô's formula to the function $g(t,r) = e^{at}r$. □

Proposition 4.5 For the solution process (4.7):

$$\begin{aligned} E[r(t)] &\to F(0,t) + \sigma^2/2a^2 \quad \text{as} \quad t \to \infty, \quad \text{and} \\ Var[r(t)] &\to \sigma^2/2a \quad \text{as} \quad t \to \infty. \end{aligned}$$

Proof Taking the expectation of the process (4.7)

$$
\begin{aligned}
E[r(t)] &= r_0 e^{-at} + \alpha(t) \\
&= r_0 e^{-at} + F(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 \\
\lim_{t\to\infty} E[r(t)] &= F(0,t) + \frac{\sigma^2}{2a^2}
\end{aligned}
$$

and for its second moment

$$
\begin{aligned}
E[r^2(t)] &= \frac{\sigma^2}{2a} + \alpha^2(t) + e^{-2at}\left(r_0^2 - \frac{\sigma^2}{2a}\right) + 2e^{-at} r_0 \alpha(t) \\
\lim_{t\to\infty} E[r^2(t)] &= \frac{\sigma^2}{2a} + F^2(0,t) + 2F(0,t)\frac{\sigma^2}{2a^2} + \frac{\sigma^4}{4a^4}. \quad \square
\end{aligned}
$$

Theorem 4.4 The solution process of equation (4.5) is exponentially mean square stable.

Proof Considering the difference of solutions, it follows that

$$
E|r(t) - r^*(t)|^2 = E|r_0 - r_0^*|^2 \exp\{-2at\}. \quad \square
$$

Hull-White: An extension

This model considers a time-dependent reversion level property while the volatility term is assumed to be proportional to the interest rate level. The extension of the Hull-White model for the interest rate process $r(t)$ given by the SDE:

$$
\begin{aligned}
dr(t) &= [\theta(t) - ar(t)]dt + \sigma r(t) dW(t), \quad t > 0, \qquad (4.8) \\
r(0) &= r_0,
\end{aligned}
$$

where r_0, a and σ are positive constants and $\theta(t)$ is a nonrandom positive function of the time variable t.

Proposition 4.6 The process satisfying the stochastic integral equation

$$r(t) = e^{-at} r_0 + e^{-at} \int_0^t e^{as} \theta(s) ds + e^{-at} \int_0^t e^{as} \sigma r(s) dW(s) \tag{4.9}$$

is the solution of the SDE (4.8).

Proof It follows by applying Itô's formula to the function $g(t, r) = e^{at} r$. $\square$

Proposition 4.7 For the solution process (4.9), its expectation can be obtained as

$$E[r(t)] = e^{-at} r_0 + e^{-at} \int_0^t e^{as} \theta(s) ds$$

and its variance can be expressed as

$$Var[r(t)] = e^{-2at} \int_0^t e^{2as} \sigma^2 r^2(s) ds.$$

Proof We take expectation and variance to the solution process (4.9) as follows:

$$\begin{aligned} E[r(t)] &= E\Big[e^{-at} r_0 + e^{-at} \int_0^t e^{as} \theta(s) ds + e^{-at} \int_0^t e^{as} \sigma r(s) dW(s)\Big] \\ &= e^{-at} r_0 + e^{-at} \int_0^t e^{as} \theta(s) ds, \end{aligned}$$

$$\begin{aligned} Var[r(t)] &= Var\Big[e^{-at} r_0 + e^{-at} \int_0^t e^{as} \theta(s) ds + e^{-at} \int_0^t e^{as} \sigma r(s) dW(s)\Big] \\ &= Var\Big[e^{-at} \int_0^t e^{as} \sigma r(s) dW(s)\Big] \\ &= e^{-2at} \int_0^t e^{2as} \sigma^2 r^2(s) ds. \quad \square \end{aligned}$$

Theorem 4.5 The solution process of equation (4.8) is exponentially mean square stable provided $a > \sigma^2$.

Proof Considering the difference of solutions, it follows proceeding as in Proposition 4.2:

$$E|r(t)-r^*(t)|^2 \leq 2E|r_0-r_0^*|^2 \exp\{-2(a-\sigma^2)t\}. \quad \square$$

4.1.4 Black-Derman-Toy

Black, Derman and Toy, 1990, proposed a binomial-tree approach for a lognormal short-rate process see Hull, 2012, p. 693.

Black, Derman and Toy interest rate model is described by the following SDE:

$$\begin{aligned} d\ln r(t) &= [h(t)-a(t)\ln r(t)]dt+\sigma(t)dW(t), \quad t>0, \\ r(0) &= r_0. \end{aligned}$$

This model imposes the following relationship between the volatility parameter $\sigma(t)$ and the reversion rate parameter $a(t)$:

$$a(t) = -\frac{\sigma'(t)}{\sigma(t)} \tag{4.10}$$

A version of this model, often used in practice, is when $\sigma(t)$ is constant. So there is no mean reversion and the Black-Derman-Toy interest rate model reduces to

$$\begin{aligned} d\ln r(t) &= h(t)dt+\sigma dW(t), \quad t>0, \\ r(0) &= r_0. \end{aligned}$$

or,

$$\begin{aligned} dr(t) &= h(t)r(t)dt+\sigma r(t)dW(t), \quad t>0, \\ r(0) &= r_0. \end{aligned} \tag{4.11}$$

The properties obtained in this section appear to be new.

Theorem 4.6 The solution process of the Black-Derman-Toy interest rate model (4.11), is

$$r(t) \quad = \quad r_0 \exp\Big\{\int_0^t h(s)ds - \frac{1}{2}\sigma^2 t + \sigma W(t)\Big\}. \tag{4.12}$$

Proof We find the solution process to (4.11) by application of Itô's formula taking the function $g(t,r) = \ln r$. Then

$$\begin{aligned}
d\Big[\ln r(t)\Big] &= \frac{\partial g}{\partial t}(t,r(t))dt + \frac{\partial g}{\partial r}(t,r(t))dr(t) + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}(t,r(t))dr(t)dr(t) \\
&= [0]dt + \frac{1}{r(t)}dr(t) + \frac{1}{2}\Big[-\frac{1}{r^2(t)}\Big]dr(t)dr(t) \\
&= \frac{1}{r(t)}[h(t)r(t)dt + \sigma r(t)dW(t)] - \frac{1}{2}\frac{1}{r^2(t)}[h(t)r(t)dt + \sigma r(t)dW(t)]^2 \\
&= \Big[h(t) - \frac{1}{2}\sigma^2\Big]dt + \sigma dW(t). \qquad (4.13)
\end{aligned}$$

Integrating equation (4.14) from 0 to t yields

$$\begin{aligned}
\ln r(t) - \ln r(0) &= \int_0^t h(s)ds - \frac{1}{2}\sigma^2 \int_0^t dt + \sigma \int_0^t dW(t) \\
\ln\Big[\frac{r(t)}{r_0}\Big] &= \int_0^t h(s)ds - \frac{1}{2}\sigma^2 t + \sigma W(t).
\end{aligned}$$

Thus, the desired result follows. □

Theorem 4.7 For the solution process (4.13) of the Black-Derman-Toy interest rate model

$$E[r(t)] = r_0 \exp\Big\{\int_0^t h(s)ds\Big\},$$

and

$$Var[r(t)] = r_0^2 \exp\Big\{2\int_0^t h(s)ds\Big\}\Big[\exp\{\sigma^2 t\} - 1\Big].$$

Proof For the first moment of the solution process of the Black-Derman-Toy interest rate model

(4.11),

$$
\begin{aligned}
E[r(t)] &= E\Big[r_0 \exp\Big\{\int_0^t h(s)ds - \frac{1}{2}\sigma^2 t + \sigma W(t)\Big\}\Big] \\
&= r_0 \exp\Big\{\int_0^t h(s)ds\Big\}.
\end{aligned}
$$

For the second moment we have

$$
\begin{aligned}
E[r^2(t)] &= E\Big[r_0^2 \exp\Big\{2\int_0^t h(s)ds\Big\} - \sigma^2 t + 2\sigma W(t)\Big] \\
&= r_0^2 \exp\Big\{2\int_0^t h(s)ds + \sigma^2 t\Big\}. \quad \square
\end{aligned}
$$

4.1.5 Black-Karasinski

Black and Karasinski, 1991, proposed an extension of the Black-Derman-Toy interest rate model where this model is no longer required to fulfill equation (4.10) so the reversion rate and volatility can be determined independently of each other.

The properties obtained in this section appear to be new.

Therefore, the Black-Karasinski interest rate model is described by the following SDE:

$$
\begin{aligned}
d\ln r(t) &= [h(t) - a(t)\ln r(t)]dt + \sigma(t)dW(t), \quad t > 0, \\
r(0) &= r_0.
\end{aligned}
$$

which is the same Black-Derman-Toy interest rate model but without a restriction. For our purposes we take the case $a = 0$ in order to address a functional form suggested by Björk, 2009, so that the Black-Karasinski model becomes

$$
\begin{aligned}
d\ln r(t) &= h(t)dt + \sigma(t)dW(t), \quad t > 0, \\
r(0) &= r_0.
\end{aligned}
$$

or,

$$\begin{aligned} dr(t) &= h(t)r(t)dt + \sigma(t)r(t)dW(t), \quad t > 0, \\ r(0) &= r_0. \end{aligned} \tag{4.14}$$

Theorem 4.8 The solution process of the Black-Karasinski interest rate model (4.15), is

$$r(t) = r_0 \exp\Big\{\int_0^t [h(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW(s)\Big\}. \tag{4.15}$$

Proof We find the solution process to (4.15) by application of Itô's formula taking the function $g(t,r) = \ln r$. Then

$$\begin{aligned} d\ln r(t) &= \frac{\partial g}{\partial t}(t, r(t))dt + \frac{\partial g}{\partial r}(t, r(t))dr(t) + \frac{1}{2}\frac{\partial^2 g}{\partial r^2}(t, r(t))dr(t)dr(t) \\ &= [0]dt + \frac{1}{r(t)}dr(t) + \frac{1}{2}\Big[-\frac{1}{r^2(t)}\Big]dr(t)dr(t) \\ &= \frac{1}{r(t)}[h(t)r(t)dt + \sigma(t)r(t)dW(t)] - \frac{1}{2}\frac{1}{r^2(t)}[h(t)r(t)dt + \sigma(t)r(t)dW(t)]^2 \\ &= \Big[h(t) - \frac{1}{2}\sigma^2(t)\Big]dt + \sigma(t)dW(t). \end{aligned} \tag{4.16}$$

Integrating equation (4.17) from 0 to t yields

$$\begin{aligned} \ln r(t) - \ln r(0) &= \int_0^t [h(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW(s) \\ \ln\Big[\frac{r(t)}{r_0}\Big] &= \int_0^t [h(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW(s). \end{aligned}$$

Thus, the desired result follows. □

Theorem 4.9 For the solution process (4.16) of the Black-Karasinski interest rate model

$$E[r(t)] = r_0 \exp\Big\{\int_0^t h(s)ds\Big\},$$

and

$$Var[r(t)] = r_0^2 \exp\Big\{2\int_0^t h(s)ds\Big\}\Big\{\exp\Big\{\int_0^t \sigma^2(s)ds\Big\} - 1\Big\}.$$

Proof For the first moment of the solution process of the Black-Karasinski interest rate model (4.15)

$$\begin{aligned} E[r(t)] &= r_0 E \exp\Big\{\int_0^t [h(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW(s)\Big\}. \\ &= r_0 \exp\Big\{\int_0^t h(s)ds\Big\}. \end{aligned}$$

For the second moment we have

$$\begin{aligned} E[r^2(t)] &= r_0^2 E \exp\Big\{2\int_0^t [h(s) - \frac{1}{2}\sigma^2(s)]ds + 2\int_0^t \sigma(s)dW(s)\Big\} \\ &= r_0^2 \exp\Big\{2\int_0^t h(s)ds + \int_0^t \sigma^2(s)ds\Big\}. \quad \square \end{aligned}$$

4.1.6 Dothan

After assuming a constant market price of risk, the Dothan interest rate model takes a continuous-time version of the Rendleman and Barter, 1980, model. The interest rate is always positive for each t considering a lognormal distribution.

The properties obtained are known for stocks given for the Black-Scholes-Merton equation, 1973. However, we obtain similar properties for the interest rate process.

We consider the following form of the Dothan model for the interest rate process $r(t)$:

$$\begin{aligned} dr(t) &= ar(t)dt + \sigma r(t)dW(t), \quad t > 0, \\ r(0) &= r_0. \end{aligned} \tag{4.17}$$

where a is a real constant and r_0 and σ are positive constants.

A well-known solution of this model is given by Svishchuk and Kalemanova, 2000:

$$r(t) \quad = \quad r_0 \exp\Big\{(a - \frac{1}{2}\sigma^2)t + \sigma W(t)\Big\}. \tag{4.18}$$

Proposition 4.8 For the solution process (4.19) of the Dothan interest rate model (4.18),

$$\begin{aligned} E[r(t)] &= r_0 \exp\{at\}, \quad \text{and} \\ Var[r(t)] &= r_0^2 \exp\{2at\}[\exp\{\sigma^2 t\} - 1]. \end{aligned}$$

Proof For the first moment of (4.19), we have

$$\begin{aligned} E|r(t)| &= r_0 \exp\Big\{\Big(a - \frac{1}{2}\sigma^2\Big)t + \sigma W(t)\Big\} \\ &= r_0 \exp\{at\}. \end{aligned}$$

For the second moment we have

$$\begin{aligned} E|r(t)|^2 &= r_0^2 E \exp\Big\{2\Big(a - \frac{1}{2}\sigma^2\Big)t + 2\sigma W(t)\Big\} \\ &= r_0^2 \exp\{2at + \sigma^2 t\}. \quad \square \end{aligned}$$

Proposition 4.9 For the solution process (4.19),

$$\lim_{t\to\infty} E[r(t)] = \begin{cases} 0 & \text{if } a < 0 \\ r_0 & \text{if } a = 0 \\ +\infty & \text{if } a > 0 \end{cases}$$

and

$$\lim_{t\to\infty} Var[r(t)] = +\infty, \quad a \geq 0.$$

Proof It can be easily shown from expressions for $E[r(t)]$ and $Var[r(t)]$ in Proposition 4.8. $\square$

Proposition 4.10 If $a < -\sigma^2$, then the trivial solution of equation (4.18) given in (4.19) is exponentially mean square stable.

Proof Let $r^*(t)$ be the solution of (4.18) satisfying $r^*(0) = r_0^*$. Then

$$r(t) - r^*(t) = (r_0 - r_0^*)\exp\Big\{(a - \frac{1}{2}\sigma^2)t + \sigma W(t)\Big\}.$$

Taking expectations on both sides of the equation, we obtain

$$E|r(t) - r^*(t)|^2 = E|r_0 - r_0^*|^2 E\exp\Big\{2(a - \frac{1}{2}\sigma^2)t + 2\sigma W(t)\Big\}.$$

Thus

$$E|r(t) - r^*(t)|^2 = E|r_0 - r_0^*|^2 \exp\Big\{2(a + \sigma^2)t\Big\}, \quad t \geq 0. \quad \square$$

4.2 Interest Rate Models with Poisson Jumps

In this Section, we consider some of the interest rate models with Poisson jumps.

4.2.1 Vasicek

Asssume that the Vasicek model of the interest rate process is continuous on time intervals $[\tau_i, \tau_{i+1}]$, $i = 1, 2, \ldots$. At random times τ_i the interest rate jumps:

$$r_{\tau_i} = (1 + U_i) r_{\tau_i -}.$$

The number of jumps on the interaval $[0, t]$, denoted by $N(t)$, is assumed to be a Poisson process with intensity λ. The jumps $(U_i)_{i \geq 0}$ for a sequence of i.i.d. random variables assuming values in $(-1, \infty)$.

According to Svishchuk and Kalemanova, 2000, the Vasicek interest rate process with compound Poisson jumps can be represented as

$$r(t) = \prod_{i=1}^{N(t)} (1 + U_i)\Big[\frac{\alpha}{\beta} + \Big(r_0 - \frac{\alpha}{\beta}\Big)e^{-\beta t} + +\sigma e^{-\beta t}\int_0^t e^{\beta s} dW(s)\Big]. \tag{4.19}$$

Note that equation (4.20) can be rewritten as

$$r(t) = \prod_{i=1}^{N(t)} (1+U_i)\Big[e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1-e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s} dW(s)\Big]. \tag{4.20}$$

where the continuous part of this jump process coincides with expression (3.7) in Chapter 3.

Theorem 4.10 The solution (4.21) of the Vasicek interest rate model with jumps satisfies

$$E|r(t)|^p \leq e^{\lambda(E[1+U_1]^p-1)t} 3^{p-1}\Big\{E|r_0|^p + \frac{\alpha^p}{\beta^p} + \Big[\frac{1}{2}p(p-1)\Big]^{p/2} T^{p/2-1}\frac{\sigma^p}{\beta p}(e^{\beta p T}-1)\Big\} \tag{4.21}$$

for $p \geq 2$ and for each $t \in [0, T]$.

Proof We observe that

$$E\Big|\prod_{i=1}^{N(t)}(1+U_i)\Big|^p = E\Big|\prod_{i=1}^{N(t)}(1+U_i)^p\Big|.$$

An application of Lemma 2.2 yields

$$E\Big|\prod_{i=1}^{N(t)}(1+U_i)^p\Big| = e^{\lambda(E[1+U_1]^p-1)t}.$$

Next, taking expectation in (4.21), and following the proof of Theorem 4.1, we obtain the desired result. □

4.2.2 Cox-Ingersoll-Ross

In this subsection, we follow the approach considered by Svishchuk and Kalemanova, 2000. We do this in two steps.

In the first step, we consider the continuous part of the jump process, equation (3.9). Denote by $B(t)$ a one dimensional Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $B(0) = 0$, $r_0 \in R$, $V(0) = r_0 - \frac{\alpha}{\beta}$, and $\tilde{V}(s) = V(0) + B(\varphi_s^{-1})$ for an increasing function $\varphi(s)$ such that

$$\varphi_t = \int_0^t a^{-2}(\varphi_s, \tilde{V}(s))\,ds, \tag{4.22}$$

where

$$a(s, \tilde{V}(s)) = \sigma e^{\beta s} \sqrt{e^{-\beta s}\tilde{V}(s) + \frac{\alpha}{\beta}}.$$

Proposition 4.11 [Svishchuk and Kalemanova, 2000] The process

$$r(t) = \frac{\alpha}{\beta} + e^{-\beta t} V(0) + e^{-\beta t} B(\varphi_t^{-1}) \tag{4.23}$$

is a solution of equation the CIR model (3.9).

Note that the solutions given by the above Proposition 4.11, and equation (3.12) are the same. To show this, consider the transformation

$$V(t) = e^{\beta t}\Big(r(t) - \frac{\alpha}{\beta}\Big).$$

Applying Itô's formula we get

$$d[V(t)] = \sigma e^{\beta t} \sqrt{e^{-\beta t} V(t) + \frac{\alpha}{\beta}} dW(t). \tag{4.24}$$

Moreover, changing the time in the Wiener process, we obtain the following solution of equation (4.25)

$$V(t) = V(0) + B(\varphi_s^{-1}) = \tilde{V}(t), \tag{4.25}$$

where

$$B(\varphi_s^{-1}) = \sigma \int_0^t e^{\beta s} \sqrt{r(s)} dW(s).$$

In the second step, consider the following CIR interest rate process with random jumps as in Svishchuk and Kalemanova, 2000,

$$r(t) = \prod_{i=1}^{N(t)} (1 + U_i) \Big[\frac{\alpha}{\beta} + e^{-\beta t}\Big(r_0 - \frac{\alpha}{\beta}\Big) + e^{-\beta t} \sigma \int_0^t e^{\beta s} \sqrt{r(s)} dW(s) \Big]. \tag{4.26}$$

The following theorem generalizes a previous result by Govindan and Acosta, 2008.

Theorem 4.11 For the CIR interest rate model with the compound Poisson jump (4.27), we have

$$\limsup_{t\to\infty} E|r(t)-r^*(t)|^2 \le e^{\lambda(E[1+U_1]^2-1)t}\,\frac{\sigma^2\alpha}{\beta^2}.$$

Proof For our purposes, we rewrite (4.27) as

$$\begin{aligned} X(t) &= r(t)e^{\beta t} \\ &= \prod_{i=1}^{N(t)}(1+U_i)\Big[r_0+\frac{\alpha}{\beta}\Big(e^{\beta t}-1\Big)+\sigma\int_0^t e^{\beta s}\sqrt{r(s)}dW(s)\Big]. \end{aligned} \tag{4.27}$$

Taking expectations on both sides of (4.28), we get

$$\begin{aligned} E[X(t)] &= E\Big\{\prod_{i=1}^{N(t)}(1+U_i)\Big[r_0+\frac{\alpha}{\beta}\Big(e^{\beta t}-1\Big)+\sigma\int_0^t e^{\beta s}\sqrt{r(s)}dW(s)\Big]\Big\} \\ &= e^{\lambda(E[1+U_1]-1)t}\Big[r_0+\frac{\alpha}{\beta}\Big(e^{\beta t}-1\Big)\Big]. \end{aligned} \tag{4.28}$$

Next, we consider the processes $X(t)=r(t)e^{\beta t}$, and $X^*(t)=r^*(t)e^{\beta t}$, where $r(t)$ and $r^*(t)$ are solutions with the initial condition $r(0)=r_0$, $r^*(0)=r_0^*$, respectively.
Recall

$$J(t)=\prod_{i=1}^{N(t)}(1+U_i).$$

From (4.28), we have

$$\begin{aligned} |X(t)-X^*(t)|^2 &= |J(t)|^2\Big\{|r_0-r_0^*|^2+2|r_0-r_0^*|\sigma\int_0^t e^{\beta s}\Big|\sqrt{r(s)}-\sqrt{r^*(s)}\Big|dW(s) \\ &\quad+\sigma^2\int_0^t e^{\beta s}\Big|\sqrt{r(s)}-\sqrt{r^*(s)}\Big|^2 ds\Big\}. \end{aligned} \tag{4.29}$$

Taking expectations on both sides of (4.30), we get

$$
\begin{aligned}
E|X(t)-X^*(t)|^2 &= E|J(t)|^2\Big\{E|r_0-r_0^*|^2 \\
&\quad+\int_0^t \sigma^2 e^{2\beta s}E\Big|\sqrt{r(s)}-\sqrt{r^*(s)}\Big|^2 ds\Big\}. \qquad (4.30)
\end{aligned}
$$

But note that

$$
\begin{aligned}
E|J(t)|^2 &= E\Big|\prod_{i=1}^{N(t)}(1+U_i)\Big|^2 \\
&= E\Big|\prod_{i=1}^{N(t)}(1+U_i)^2\Big| \\
&= e^{\lambda(E[1+U_1]^2-1)t}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
E\Big[\sqrt{r(s)}-\sqrt{r^*(s)}\Big]^2 &= E\Big[r(s)-2\sqrt{r(s)}\sqrt{r^*(s)}+r^*(s)\Big] \\
&\le E|r(s)|+E|r^*(s)|. \qquad (4.31)
\end{aligned}
$$

For the continuous part of the jump process, equation (3.12), we note that

$$
\begin{aligned}
E|r(s)| &= r_0e^{-\beta s}+\frac{\alpha}{\beta}(1-e^{-\beta s}), \\
E|r^*(s)| &= r_0^*e^{-\beta s}+\frac{\alpha}{\beta}(1-e^{-\beta s}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
E|X(t)-X^*(t)|^2 &\le E|J(t)|^2\Big\{E|r_0-r_0^*|^2 \\
&\quad+\int_0^t \sigma^2 e^{2\beta s}\Big[E|r(s)|+E|r^*(s)|\Big]ds\Big\} \\
&\le E|J(t)|^2\Big\{E|r_0-r_0^*|^2 \\
&\quad+\int_0^t \sigma^2 e^{2\beta s}E|r(s)|ds+\int_0^t \sigma^2 e^{2\beta s}E|r^*(s)|ds\Big\}. \qquad (4.32)
\end{aligned}
$$

From (4.33), we obtain

$$\int_0^t \sigma^2 e^{2\beta s} E|r(s)|ds = \frac{1}{\beta} r_0 \sigma^2 \left(e^{\beta t} - 1\right) + \frac{\alpha\sigma^2}{2\beta^2}\left(e^{2\beta t} - 1\right) - \frac{\alpha\sigma^2}{\beta^2}\left(e^{\beta t} - 1\right),$$
$$\int_0^t \sigma^2 e^{2\beta s} E|r^*(s)|ds = \frac{1}{\beta} r_0^* \sigma^2 \left(e^{\beta t} - 1\right) + \frac{\alpha\sigma^2}{2\beta^2}\left(e^{2\beta t} - 1\right) - \frac{\alpha\sigma^2}{\beta^2}\left(e^{\beta t} - 1\right).$$

Hence, from (4.33), we have

$$\begin{aligned} E|r(t) - r^*(t)|^2 \quad \leq \quad & E|J(t)|^2 \Big\{ E|r_0 - r_0^*|^2 e^{-2\beta t} \\ & + \frac{\sigma^2}{\beta}\left(1 - e^{-\beta t}\right)(r_0 - r_0^*) e^{-\beta t} \\ & + \frac{\alpha\sigma^2}{\beta^2}\left(1 - e^{-2\beta t}\right) - 2\frac{\alpha\sigma^2}{\beta^2}\left(1 - e^{-\beta t}\right) e^{-\beta t} \Big\}. \end{aligned}$$

from which the desired result follows. □

4.2.3 Black-Derman-Toy

Consider the Black-Derman-Toy interest rate model with compensated compound Poisson jumps:

$$\begin{aligned} dr(t) &= h(t)r(t)dt + \sigma r(t)dW(t) + r(t-)d[Q(t) - \lambda\gamma t], \quad t > 0, \\ r(0) &= r_0, \end{aligned}$$

where $h(t)$ is a deterministic function and σ, λ and γ are nonnegative constants.

Note that this equation can be re-written as

$$\begin{aligned} dr(t) &= [h(t) - \lambda\gamma]r(t)dt + \sigma r(t)dW(t) + r(t-)dQ(t), \quad t > 0, \\ r(0) &= r_0. \end{aligned} \tag{4.33}$$

Theorem 4.12 The solution to (4.34) is given by

$$r(t) = \prod_{i=1}^{N(t)} (1 + U_i) r_0 \exp\Big\{ \int_0^t h(s)ds - (\lambda\gamma + \frac{1}{2}\sigma^2)t + \sigma W(t) \Big\}. \tag{4.34}$$

Proof Following Shreve, 2004, p. 513, we show that (4.35) satisfies the SDE (4.34). For the continuous part of the jump process define

$$X(t) = r_0 \exp\Big\{\int_0^t h(s)ds - \Big(\lambda\gamma + \frac{1}{2}\sigma^2\Big)t + \sigma W(t)\Big\},$$

and for the pure jump process define

$$J(t) = \prod_{i=1}^{N(t)} (1 + U_i).$$

Next, we show that $r(t) = J(t)X(t)$ is a solution to the SDE (4.34). The Itô formula (Theorem 2.6) for a continuous part of the process says that

$$dX(t) = [h(t) - \lambda\gamma]X(t)dt + \sigma X(t)dW(t). \tag{4.35}$$

When considering the i^{th}-jump, $J(t) = J(t-)(1 + U_i)$, we get

$$\begin{aligned} \Delta J(t) &= J(t) - J(t-) \\ &= J(t-)U_i \\ &= J(t-)\Delta Q(t). \end{aligned}$$

The equation $\Delta J(t) = J(t-)\Delta Q(t)$ also holds at non jump times with both sides equal to zero. Hence

$$dJ(t) = J(t-)dQ(t). \tag{4.36}$$

Now from Itô's product rule (Theorem 2.9) for a jump processes,

$$\begin{aligned} r(t) &= X(t)J(t) \\ &= r_0 + \int_0^t X(s-)dJ(s) + \int_0^t J(s)dX(s) + [X, J](t). \end{aligned} \tag{4.37}$$

For a pure jump process J and a continuous process X, $[X, J](t) = 0$. Substituting (4.36) and

(4.37) into (4.38), we obtain

$$\begin{aligned} r(t) &= X(t)J(t) \\ &= r_0 + \int_0^t X(s-)J(s-)dQ(s) + \int_0^t [h(s) - \lambda\gamma]X(s)J(s)ds \\ &\quad + \sigma \int_0^t X(s)J(s)dW(s), \end{aligned}$$

which expressed in a differential form gives

$$\begin{aligned} dr(t) &= d\Big[X(t)J(t)\Big] \\ &= X(t-)J(t-)dQ(t) + [h(t) - \lambda\gamma]X(t)J(t)dt + \sigma X(t)J(t)dW(t) \\ &= r(t-)dQ(t) + [h(t) - \lambda\gamma]r(t)dt + \sigma r(t)dW(t), \end{aligned}$$

which is (4.34). $\square$

Corollary 4.1 The Black-Derman-Toy interest model with a compound Poisson process $Q(t)$ jumps, the solution process to (4.34) reduces to

$$r(t) = r_0 \prod_{i=1}^{N(t)} (1 + U_i) \exp\Big\{\int_0^t h(s)ds - \frac{1}{2}\sigma^2 t + \sigma W(t)\Big\}.$$

Theorem 4.13 For the solution process (4.35),

$$E[r(t)] = r_0 \exp\{\int_0^t h(s)ds\},$$

and

$$Var[r(t)] = r_0^2 \exp\{2\int_0^t h(s)ds\}[\exp\{\lambda(E[1+U_1]^2 - 1)t - 2\lambda\gamma t + \sigma^2 t\} - 1].$$

Proof For the first moment we have

$$
\begin{aligned}
E[r(t)] &= E\Big[\prod_{i=1}^{N(t)}(1+U_i)r_0\exp\Big\{\int_0^t h(s)ds-(\lambda\gamma+\frac{1}{2}\sigma^2)t+\sigma W(t)\Big\}\Big] \\
&= r_0\exp\{\lambda(E[1+U_1]-1)t\}E\Big[\exp\Big\{\int_0^t h(s)ds-(\lambda\gamma+\frac{1}{2}\sigma^2)t+\sigma W(t)\Big\}\Big] \\
&= r_0\exp\Big\{\int_0^t h(s)ds\Big\}.
\end{aligned}
$$

For the second moment we have

$$
\begin{aligned}
E[r^2(t)] &= r_0^2E\Big[\prod_{i=1}^{N(t)}(1+U_i)^2\Big]E\exp\Big\{2\int_0^t h(s)ds-2\lambda\gamma t-\sigma^2t+2\sigma W(t)\Big\} \\
&= r_0^2\exp\{\lambda(E[1+U_1]^2-1)t\}E\exp\Big\{2\int_0^t h(s)ds-2\lambda\gamma t-\sigma^2t+2\sigma W(t)\Big\} \\
&= r_0^2\exp\{\lambda(E[1+U_1]^2-1)t\}\exp\Big\{2\int_0^t h(s)ds-2\lambda\gamma t+\sigma^2t\Big\}.
\end{aligned}
$$

Hence, the desired result follows. □

4.2.4 Black-Karasinski

Consider the Black-Karasinski interest rate model with compensated compound Poisson process jumps

$$
\begin{aligned}
dr(t) &= h(t)r(t)dt+\sigma(t)r(t)dW(t)+r(t-)d[Q(t)-\lambda\gamma t], \quad t>0, \\
r(0) &= r_0,
\end{aligned}
$$

where $h(t)$ and $\sigma(t)$ are deterministic functions and λ and γ are nonnegative constants. Note that this equation can be re-written as

$$
\begin{aligned}
dr(t) &= [h(t)-\lambda\gamma]r(t)dt+\sigma(t)r(t)dW(t)+r(t-)dQ(t), \quad t>0, \\
r(0) &= r_0.
\end{aligned}
\tag{4.38}
$$

Theorem 4.14 The solution to (4.39) is given by

$$r(t) = r_0 \prod_{i=1}^{N(t)} (1+U_i) \exp\Big\{\int_0^t [h(s) - \lambda\gamma - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW(s)\Big\}. \qquad (4.39)$$

Proof Following Shreve, 2004, p. 513, we show that (4.40) satisfies the SDE (4.39) as before. For the continuous part of the jump process define

$$X(t) = r_0 \exp\Big\{\int_0^t [h(s) - \lambda\gamma - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW(s)\Big\},$$

and for the pure jump process define

$$J(t) = \prod_{i=1}^{N(t)} (1+U_i).$$

Next show that $r(t) = J(t)X(t)$ is a solution to the SDE (4.39). The Itô's formula for a continuous process yields

$$dX(t) = [h(t) - \lambda\gamma]X(t)dt + \sigma(t)X(t)dW(t). \qquad (4.40)$$

When considering the i^{th}-jump, $J(t) = J(t-)(1+U_i)$,

$$\begin{aligned}\Delta J(t) &= J(t) - J(t-)\\ &= J(t-)U_i\\ &= J(t-)\Delta Q(t).\end{aligned}$$

The equation $\Delta J(t) = J(t-)\Delta Q(t)$ also holds at non jump times with both sides equal to zero. Hence

$$dJ(t) = J(t-)dQ(t). \qquad (4.41)$$

Now from Itô's product rule for a jump processes,

$$\begin{aligned} r(t) &= X(t)J(t) \\ &= r_0 + \int_0^t X(s-)dJ(s) + \int_0^t J(s)dX(s) + [X,J](t). \end{aligned} \tag{4.42}$$

For a pure jump process J and a continuous process X, $[X,J](t) = 0$. Substituting (4.41) and (4.42) into (4.43), we obtain

$$\begin{aligned} r(t) &= X(t)J(t) \\ &= r_0 + \int_0^t X(s-)J(s-)dQ(s) + \int_0^t [h(s) - \lambda\gamma]X(s)J(s)ds \\ &\quad + \int_0^t \sigma(s)X(s)J(s)dW(s), \end{aligned}$$

which expressed in a differential form gives

$$\begin{aligned} dr(t) &= d\Big[X(t)J(t)\Big] \\ &= X(t-)J(t-)dQ(t) + [h(t) - \lambda\gamma]X(t)J(t)dt + \sigma(t)X(t)J(t)dW(t) \\ &= r(t-)dQ(t) + [h(t) - \lambda\gamma]r(t)dt + \sigma(t)r(t)dW(t) \end{aligned}$$

which is (4.39). □

Corollary 4.2 The Black-Karasinski interest rate model with compound Poisson process $Q(t)$ jumps, the solution process to (4.39) reduces to

$$r(t) = r_0 \prod_{i=1}^{N(t)} (1+U_i) \exp\Big\{\int_0^t [h(s) - \frac{1}{2}\sigma^2(t)]ds + \int_0^t \sigma(s)dW(s)\Big\}. \tag{4.43}$$

Theorem 4.15 For the solution process (4.40),

$$E[r(t)] = r_0 \exp\Big\{\int_0^t h(s)ds\Big\},$$

and

$$Var[r(t)] = r_0^2 \exp\left\{2\int_0^t h(s)ds\right\}\left[\exp\{\lambda(E[1+U_1]^2-1)t-2\lambda\gamma t+\int_0^t \sigma^2(s)ds\}-1\right].$$

Proof For the first moment we have

$$\begin{aligned}
E|r(t)| &= r_0 E\Big|\prod_{i=1}^{N(t)}(1+U_i)\Big| E\exp\Big\{\int_0^t [h(s)-(\lambda\gamma+\frac{1}{2}\sigma^2(s)]ds+\int_0^t \sigma(s)dW(s)\Big\}\\
&= r_0\exp\{\lambda(E[1+U_1]-1)t\}E\exp\Big\{\int_0^t [h(s)-(\lambda\gamma+\frac{1}{2}\sigma^2(s)]ds+\int_0^t \sigma(s)dW(s)\Big\}\\
&= r_0\exp\Big\{\int_0^t h(s)ds\Big\}.
\end{aligned}$$

For the second moment we have

$$\begin{aligned}
E|r(t)|^2 &= r_0^2 E\Big|\prod_{i=1}^{N(t)}(1+U_i)^2\Big| E\exp\Big\{2\int_0^t [h(s)-(\lambda\gamma+\frac{1}{2}\sigma^2(s)]ds+2\int_0^t \sigma(s)dW(s)\Big\}\\
&= r_0^2\exp\{\lambda(E[1+U_1]^2-1)t\}\exp\Big\{2\int_0^t h(s)ds-2\lambda\gamma t+\int_0^t\sigma^2(s)ds\Big\}.
\end{aligned}$$

Hence, the desired result follows. □

Summary

In this thesis the solution process and some probabilistic properties for the following interest rate models were studied: Vasicek, Cox-Ingersoll-Ross, Hull-White, Extended Hull-White, Hull-White: A CIR style extension, Hull-White: A special case, Hull-White: An extension, Brennan-Schwartz, Exponential Vasicek, Dothan, a Generalized interest rate model, Extended- Exponential Vasicek, Mercurio-Moraleda, Black-Derman-Toy, and Black-Karasinski. The Brownian bridge was also studied. Some interest rate models with jumps were also studied such as: Vasicek with the compound Poisson jump process, Cox-Ingersoll-Ross with the compound Poisson jump process, Black-Derman-Toy with the compensated compound Poisson jump process, and Black-Karasinski with the compensated compound Poisson jump process.

In Chapter 3, an alternative method to Itô's formula called as a linear ODE method was introduced to obtain the solution of linear interest rate models. The solution process to these models studied in this chapter was obtained by the application of the linear ODE method and also by Itô's formula. Note that the solution process obtained by both approaches was the same. Precisely, the solution process was obtained for the interest rate models such as Vasicek in Theorem 3.1, Cox-Ingersoll-Ross (CIR) in Theorem 3.2, Hull-White in Theorem 3.3, extended Hull-White in Theorem 3.4, a CIR style extension of the Hull-White in Theorem 3.5, Brennan-Schwartz in Theorem 3.6, exponential Vasicek in Theorem 3.7, Dothan in Theorem 3.8, a generalized interest rate model in Theorem 3.9, Mercurio-Moraleda in Theorem 3.10, extended exponential Vasicek in Theorem 3.11, and the Brownian bridge in Theorem 3.12. A counter example was introduced to show a case where the linear ODE method was not applicable.

In Chapter 4, we first obtained the solution process by using Itô's product rule for the in-

terest rate models such as Black-Derman-Toy and Black-Karasinsk with Poisson jumps (Lévy processes) in Theorem 4.12 and Theorem 4.14, respectively. Secondly, we obtained moment properties of Vasicek in Theorem 4.1, Brennan-Schwartz in Proposition 4.1 and Theorem 4.2, Hull-White in Proposition 4.3, extension of Hull-White in Proposition 4.7, Black-Derman-Toy in Theorem 4.7, Black-Karasinski in Theorem 4.9 and Dothan in Proposition 4.8. Also, we obtained moment properties of Vasicek in Theorem 4.10, Black-Derman-Toy in Theorem 4.13 and Black-Karasinski in Theorem 4.15 with Poisson jumps. Lastly, we obtained stability properties like boundedness in probability uniformly in t of Brennan-Schwartz model in Theorem 4.3, exponential mean square stability of Brennan-Schwartz model in Proposition 4.2, Hull-White model in Theorem 4.4 and Dothan model in Proposition 4.10, and asymptotic quadratic mean of CIR with Poisson jumps in Theorem 4.11.

A comprehensive treatment of all standard interest rate models here is beyond the scope of this thesis. However, an interested reader can consult references such as Brigo and Mercurio, 2007, and Veronesi, 2011, among others.

As far as future research is concerned, we are interested in the statistical estimation of parameters of interest rate models studied in the thesis. Similarly, we are interested in simulating the stability behavior of some results obtained here. The study of models for the valuation of bonds and their probabilistic properties is also of our interest in the future. We will continue the study of more interest rate models including such models with Poisson jumps (Lévy processes).

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